TYPICAL ORBITS OF QUADRATIC POLYNOMIALS WITH A NEUTRAL FIXED POINT I: NON-BRJUNO TYPE

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ABSTRACT. We study (Lebesgue) typical *orbits* of quadratic polynomials $P_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z + z^2 : \mathbb{C} \to \mathbb{C}$, with α of *non-Brjuno* and *high return* type. This includes quadratic polynomials with positive area *Julia* set of X. Buff and A. Chératat. As a consequence, we introduce rational maps of arbitrarily large degree for which the Brjuno condition is optimal for their *linearizability*. Our technique uses the *near-parabolic renormalization* developed by H. Inou and M. Shishikura.

INTRODUCTION

The *Julia set* of

$$P_\alpha(z) := e^{2\pi\alpha\mathbf{i}}z + z^2 : \mathbb{C} \to \mathbb{C},$$

denoted by $J(P_\alpha)$, is defined as the closure of the set of *repelling* periodic orbits of P_α. It turns out that the structure of $J(P_\alpha)$, for $\alpha \in [0, 1]$, depends dramatically on the arithmetic nature of α. For $\alpha \in \mathbb{Q}$, by [DH84] there are essentially two types of orbits lying in the Fatou set of P_α (i.e. the maximal domain of normality of the iterates P_α^n). To explain the case of irrational rotations, we use the continued fraction expansion

$$\alpha = [a_1, a_2, a_3, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

with all $a_i \in \mathbb{N}$. Also, let $p_n/q_n := [a_0, a_1, \dots, a_n]$, for $n \geq 0$, denote the best rational approximants of α.

Let f be a holomorphic function defined near $z = 0$ that leaves zero fixed and has *multiplier* $f'(0) = e^{2\pi\alpha\mathbf{i}}$ with $\alpha \in [0, 1]$. The map f is called *linearizable* at $z = 0$ if there exists a holomorphic change of coordinate ϕ defined near 0 with $\phi(0) = 0$ and $\phi \circ f \circ \phi^{-1}(z) = e^{2\pi\alpha\mathbf{i}} \cdot z$. By a theorem of A. D. Brjuno [Brj71], every such germ is linearizable at 0 if α is irrational and the infinite series

$$\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n},$$

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is convergent. In the linearizable case, the maximal domain of linearization is called the *Siegel* disk of that germ. On the other hand, a celebrated work of J.-C. Yoccoz [Yoc95] showed that the above condition is sharp in the quadratic family. That is, if the above series is divergent, then the quadratic polynomial P_α is not linearizable at zero. R. Pérez-Marco in [PM93] has generalized this optimality to an open dense (structurally stable) set of polynomials. To my understanding, the great idea of hedgehogs introduced in [PM97] gives the first insight into local dynamics of non-linearizable germs.

An irrational number α is called *Brjuno* if the above series is convergent. Otherwise, it is called non-Brjuno.

In the non-linearizable case, $J(P_\alpha)$ is known to have a complicated topology. For example, it is a non locally connected subset of the plane (see [Sul85]), and by [Mañ93] the orbit of the critical point is recurrent and accumulates at the zero fixed point.

C. Peterson and S. Zakeri in [PZ04] have proved that for Lebesgue almost every $\alpha \in [0,1]$ (indeed, when $a_n = \mathcal{O}(\sqrt{n})$) $J(P_\alpha)$ is locally connected and has zero area. This gives a complete understanding of the dynamics of P_α on $J(P_\alpha)$ for these quadratics that possess a Siegel disk. However, a remarkable recent result of X. Buff and C. Chéritat [BC05] shows the existence of parameters α, both of Brjuno and non-Brjuno type, for which the Julia sets $J(P_\alpha)$ have positive area. This motivates describing the orbits of typical points, with respect to the Lebesgue measure, in the Julia sets. Here, we mainly focus on the non-linearizable case, and leave the linearizable case to a forthcoming note.

Let Irr_N, for a given integer N, denote the real numbers $\alpha = [a_1, a_2, \dots]$ with all $a_i \geq N$. The following is the first result in this direction.

Theorem A. *There exists a constant N such that for every non-Brjuno $\alpha \in Irr_N$, the orbit of (Lebesgue) almost every point in the Julia set of $P_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z + z^2$ accumulates at the fixed point $z = 0$.*

The above theorem applies to the non-linearizable examples by Buff and Chéritat and it is meaningful for every positive area Julia set satisfying the hypotheses.

The idea of the proof is to build infinitely many "gates" with the 0 fixed point on their boundaries such that almost every point in the Julia set has to go through them. To define these gates, we have used the *near-parabolic renormalization*, a slight variation of the Poincaré return map to certain domains, developed by H. Inou and M. Shishikura in [IS06]. More precisely, they have defined a class of holomorphic maps by their covering property which is invariant under this renormalization. This result of Inou-Shishikura has been part of the big project of finding quadratics with positive area Julia sets. To prove the theorem, we control the Euclidean diameter of these gates in terms of the partial sums of the above series, and conclude that the diameters go to 0 once α is non-Brjuno.

We observe, along the proof of the above theorem, that the orbit of the critical point goes through the above mentioned gates regardless of the area of the Julia set. Hence, this provides a direct dynamical proof of the Yoccoz's optimality result (on the non-linearizability) for multipliers with high return type. Our argument works for all maps in the Inou-Shishikura class and enables us to obtain the following:

Theorem B. *There exists a constant N such that if α is a non-Brjuno number in Irr_N, then P_α and every map in the Inou-Shishikura class, with the fixed point of multiplier $e^{2\pi\alpha\mathbf{i}}$ at 0, is non-linearizable at the 0 fixed point.*

In particular, there exists a Jordan domain U containing 0 such that if f is a map of the form $e^{2\pi\alpha\mathbf{i}}h(1+h)^2$, where $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map of the Riemann sphere satisfying

- *$h(0) = 0$, $h'(0) = 1$, and*
- *no critical value of h belongs to U,*

then f is not linearizable at zero.

The domain U is defined precisely in Equation (1) on Page 7. Roughly speaking, the above condition states that f has only one simple critical point in the component of $h^{-1}(U)$ containing 0 which is interacting with the fixed point at zero. The proof uses only dynamics of f on the component of $h^{-1}(U)$ containing 0.

The *post-critical* set of P_α is defined as the closure of the orbit of the critical value of P_α. By a theorem of M. Lyubich [Lyu83], the post-critical set of P_α is the measure theoretic attractor of P_α on its Julia set. That is, almost every point in $J(P_\alpha)$ eventually stays in arbitrary neighborhoods of the post-critical set. To study the (Lebesgue) measurable dynamics of P_α on its Julia set, it is useful to understand the geometry and topology of its post-critical set.

Theorem C. *There exists a constant N such that for every non-Bruno $\alpha \in Irr_N$, the post-critical set of P_α has zero area.*

To prove the theorem, we first define a nest of topological disks containing the post-critical set. The topological disks are transferred from the dynamical planes of the renormalizations of P_α to the dynamical plane of P_α. Then, given a point z in the intersection, by controlling the geometry of these topological disks, we find a sequence of balls in the complements whose radii are proportional to their distance to z. This requires an analysis involving the arithmetic of α. We should mention that this happens for a particular sequence of such radii (depending on α and z) and does not imply that the post-critical set of these maps is porous.

An immediate corollary of the above theorem is the following

Corollary D. *There exists a constant N such that almost every point in the Julia set of P_α with a non-Brjuno $\alpha \in Irr_N$ is non-recurrent. In particular, there is no finite absolutely continuous invariant measure on the Julia set.*

By controlling the *changes of coordinates* between different levels of renormalizations, we show that the orbit of the critical value forms ε-chains in the post-critical set. This implies the following.

Theorem E. *There exists a constant N such that for every $\alpha \in Irr_N$ the post-critical set of P_α is connected.* [1]

[1] In a private communication, S. Zakeri, based on the theory of hedgehogs introduced by Pérez-Marco, proved the connectivity of the post-critical set of all quadratic polynomials with an irrationally neutral fixed point.

A main tool in our analysis is estimating the changes of coordinates (perturbed Fatou coordinates) between different levels of renormalizations. Such estimates, based on the theory of univalent maps, are given mainly in [IS06] for the *parabolic* maps (see also [Shi98]). The change of coordinates depends continuously, in the compact open topology, on the map. Therefore, they may be used to obtain some estimates for the perturbed maps. However, the continuity is in a too weak topology to derive formulas for the perturbed coordinates. Here, we give a different type of estimates, based on the quasi-conformal mapping techniques, for the perturbed maps and analyze how they behave when the maps converge to the parabolic ones. I believe the tools introduced here can be used to answer fine scale questions on the geometry of the post-critical set. There are also very fine estimates, based on comparing with (the time one map of) a suitable vector field, given in [BC06]. However, these estimates work on a smaller domain and cannot be used here.

Remark. There has been a particular effort to keep all the constants N in the above results the same as the one obtained in [IS06]. Although, some of the arguments could be to some extent simplified by requiring a larger constant. Indeed, successive Inou-Shishikura renormalizations of P_α produce a sequence of maps with a fixed covering property, a critical point in their domain of definition, and a uniform lower bound on their non-linearities. However, we do not use the particular covering property here. The same results hold if the successive renormalization of P_α produces a sequence of maps with a critical point in their domain of definition and a uniform lower bound on their non-linearities.

Organization of the paper. Section 1 starts with basic definitions in holomorphic dynamics, Koebe distortion theorem, and an example of how this theorem is used through our work. It continuous with introducing the Inou-Shishikura class of maps and the near-parabolic renormalization on this class. We do not touch the renormalization of parabolic maps here and only recall part of the results in [IS06] on the renormalization of perturbed maps relevant to our problem. In section 2 we prove Theorems A and B assuming two technical lemmas on the perturbed Fatou coordinates. In section 3 we prove some estimates for these coordinates and derive the two technical lemmas from them. This section is independent of the other parts and can be read on its on. Section 4 is devoted to proofs of Theorems C and E.

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Frequently used notations.

- $:=$ is used when a notation is defined for the first time.
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, denote the integer, rational, real, and complex numbers, respectively. $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere.
- $\mathbf{i}$ denotes the imaginary unit complex number; $\mathbf{i}^2 = -1$, and i is used as an integer index.

– $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$ denote the real part, imaginary part, and absolute value of a complex number z, respectively.
– $B(y, \delta) \subset \mathbb{C}$ denotes the ball of radius δ around y in the Euclidean metric. If there is a different metric involved we use $B_d(y, \delta)$ to denote the ball of radius δ around y in metric d.
– $\operatorname{diam}(S)$ denotes the Euclidean diameter of a set $S \subset \mathbb{C}$.
– Given a map f, f^n denotes the n times composition of f with itself.
– $\operatorname{Dom} f$, $J(f)$, and $\mathcal{PC}(f)$ denote the domain of definition, Julia set, and the post-critical set of a map f, respectively.
– $\mathcal{O}_f(z)$ and $\omega_f(z)$ denote the orbit and the limit set of a point z under a map f. When it is clear what map is involved we drop the subscript f.
– Irr_N denotes the set of irrational numbers $[a_1, a_2, a_3, \cdots]$ with all $a_i \geq N$.
– Almost every, or a.e. for short, refers to almost every point with respect to the Lebesgue measure.
– Univalent map refers to a one to one holomorphic map.
– Given $g \colon \operatorname{Dom} g \to \mathbb{C}$, with only one critical point in its domain of definition, cp_g and cv_g denote the critical point and the critical value of g, respectively.
– $\mathbb{E}\mathrm{xp} := \frac{-4}{27} e^{-2\pi \mathbf{i} \bar{z}}$, where $\bar{z}$ denotes the complex conjugate of z.
– For every $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

1. Preliminaries and renormalization

1.1. **Post-critical set as an attractor.** Let $f \colon U \subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Given a point $z \in U$, if $f(z)$ belongs to U we can define $f^2(z) = f \circ f(z)$. Similarly, if $f^2(z)$ also belongs to U, $f^3(z)$ is defined and so on. *Orbit of* z, denoted by $\mathcal{O}(z)$, is the sequence, $z, f(z), f^2(z), \ldots$, as long as it is defined. So it may be a finite or an infinite sequence. We say that $\mathcal{O}(z)$ *eventually stays* in a given set $E \subset \hat{\mathbb{C}}$, if there exists an integer k such that for every integer $i \geq k$, $f^i(z) \in E$.

The *Fatou set* of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined as the largest open set $F(f) \subseteq \hat{\mathbb{C}}$ on which the sequense of iterates $\langle f^n \rangle_{n=0,1,\ldots}$ forms a normal family. Its complement, denoted by $J(f)$, is called the *Julia set* of f. The *post-critical* set of $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined as

$$\mathcal{PC}(f) := \bigcup_{c : f'(c)=0} \overline{\mathcal{O}(c)}.$$

Distortion of a map $f \colon U \subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the supremum of $\log(|f'(z)/f'(w)|)$, in the spherical distance, for all z and w in U, which may be finite or infinite.

We say that a simply connected domain $U \subset \mathbb{C}$, different from $\mathbb{C}$ itself, has *bounded eccentricity*, if there exists a univalent onto map $\psi \colon B(0, 1) \to U$, a uniformization, with bounded distortion. One can see that if a simply connected domain U has bounded distortion M, then ratio of radii of the smallest circle containing U, and the largest circle contained in U, is less than some constant depending only on M.

We frequently use the following distortion theorem due to Koebe [Pom75] to transfer areas under holomorphic maps.

Theorem 1.1. *(Koebe distortion theorem) Suppose that $f\colon B(0,1) \to \mathbb{C}$ is a univalent function with $f(0) = 0$, and $f'(0) = 1$. For every $z \in B(0,1)$ we have the following estimates*

(1) $\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$,
(2) $\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$,
(3) $\frac{1-|z|}{1+|z|} \leq |zf'(z)/f(z)| \leq \frac{1+|z|}{1-|z|}$.

This implies the $1/4$ theorem: the domain $f(B(0,1))$ contains $B(0,1/4)$.

The following result in [Lyu83] shows that the post-critical set of a rational map attracts the orbit of almost every point in the Julia set.

Proposition 1.2. *Let $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with $J(f) \neq \hat{\mathbb{C}}$, and V be an arbitrary neighborhood of $\mathcal{PC}(f)$. Then, the orbit of almost every point in the Julia set of f eventually stays in V.*

Here, we give a simple argument based on the Montel's normal family theorem for readers convenience. Also, we will use this approach in the next section, with iterating a renormalization operator instead of the map itself, to examine the Lebesgue measure of certain post-critical sets.

Proof. Consider the set

$$\Gamma := \{z \in J \mid \text{for infinitely many integers } k > 0, f^k(z) \notin V\}.$$

If area of Γ is not zero, let z be a Lebesgue density point of Γ. Let n_k be an increasing sequence of positive integers with $f^{n_k}(z) \notin V$, and let y be an accumulation point of the sequence $\langle f^{n_k}(z)\rangle$. As $y \notin V$, it has a definite distance δ from $\mathcal{PC}(f)$. For sufficiently large n_k, let E_{n_k} denote the component of $f^{-n_k}(B(y,\delta/2))$ containing z. As $B(y,\delta/2)$ does not intersect $\mathcal{PC}(f)$, $f^{n_k}\colon E_{n_k} \to B(y,\delta/2)$ is univalent, and in addition, its inverse has a univalent extension over the larger domain $B(y,\delta)$. By the Koebe distortion theorem, all the domains E_{n_k} have bounded eccentricity, and the maps $f^{n_k}\colon E_{n_k} \to B(y,\delta/2)$ have uniformly bounded distortion.

If E_{n_k}'s do not shrink to z as $n_k \to \infty$, their uniformly bounded eccentricity implies that E_{n_k}'s contain a ball $B(z,r)$, for some constant $r > 0$. Thus, every member of the sequence $\langle f^{n_k}\rangle$ maps $B(z,r)$ into $B(y,\delta)$. This implies that $\{f^{n_k}\}$ is a normal family, by Montel's theorem, contradicting z being in $J(f)$. Therefore, diameter of E_{n_k} tends to 0.

Also, the family E_{n_k} *shrinks regularly* to z, i.e. there exists a constant $c > 0$ such that for each E_{n_k}, there exists a round ball B with

$$E_{n_k} \subset B, \text{ and } \operatorname{area}(E_{n_k}) \geq c \cdot \operatorname{area} B.$$

As z is a Lebesgue density point of Γ (and so of J), Lebesgue's density theorem implies that,

$$\lim_{n_k \to \infty} \frac{area(E_{n_k} \bigcap \Gamma)}{area(E_{n_k})} = 1.$$

As f^{n_k}'s have bounded distortion, and Γ is f invariant, we have

$$\lim_{n_k\to\infty} \frac{area(f^{n_k}(E_{n_k}\bigcap\Gamma))}{area(f^{n_k}(E_{n_k}))} = \lim_{n_k\to\infty} \frac{area(B(y,\delta/2)\bigcap\Gamma)}{area(B(y,\delta/2))} = 1.$$

One concludes from the last equality, and that $\Gamma \subseteq J$, to get $B(y,\delta/2) \subseteq J$. This implies that $J = \hat{\mathbb{C}}$, contradicting our assumption. □

1.2. **Inou-Shishikura class and near-parabolic renormalization.**

Continued fractions: We use a slightly different type of continued fractions defined as follows. Any irrational number $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ can be written as a continued fraction of the form:

$$\alpha = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{\ddots}}}$$

where $a_n \in \mathbb{Z}$ and $\varepsilon_n = \pm 1$, for $n = 0,1,2,\dots$. For any real number $\alpha \in \mathbb{R}$, define $\|\alpha\| := \min\{|x-n| : n \in \mathbb{Z}\}$. Let $\alpha_0 = \|\alpha\|$ and a_0 be the closest integer to α, so that $\alpha = a_0 \pm \alpha_0$. For every $n \geq 0$, let $\alpha_{n+1} = \|\frac{1}{\alpha_n}\|$ and a_{n+1} be the closest integer to $\frac{1}{\alpha_n}$. Then the signs ε_n are determined by $\frac{1}{\alpha_{n-1}} = a_n + \varepsilon_n\alpha_n$. Note that α_n belongs to $(0,1/2)$ for every $n \geq 1$.

Consider a map $h\colon \mathrm{Dom}\,(h) \to \mathbb{C}$, where $\mathrm{Dom}\,(h)$ denotes domain of h. Given a compact set $K \subset \mathrm{Dom}\,(h)$ and an $\varepsilon > 0$, a neighborhood of h is defined as

$$\mathcal{N}(h,K,\varepsilon) := \{g\colon \mathrm{Dom}\,(g) \to \mathbb{C} \mid K \subset \mathrm{Dom}\,(g), \text{ and } \sup_{z\in K}|g(z)-h(z)| < \varepsilon\}.$$

By a sequence $h_n\colon \mathrm{Dom}\,(h_n) \to \mathbb{C}$ (not necessarily defined on the same set) *converges* to h we mean that given an arbitrary neighborhood of h defined as above, h_n is contained in this neighborhood for sufficiently large n.

Inou-Shishikura class: Consider the cubic polynomial $P(z) = z(1+z)^2$. This polynomial has a *parabolic* fixed point at 0, a critical point at $-1/3$ which is mapped to the critical value at $-4/27$, and another critical point at -1 which is mapped to 0. See Figure 1.

Define

$$U := P^{-1}(B(0, \frac{4}{27}e^{4\pi})) \setminus ((-\infty,-1] \cup B) \tag{1}$$

where B is the component of $P^{-1}(B(0,\frac{4}{27}e^{-4\pi}))$ containing -1.

Define the class of maps

$$\mathcal{IS} := \left\{ P\circ\varphi^{-1}\colon U_f \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi\colon U \to U_f \text{ is univalent}, \varphi(0)=0,\ \varphi'(0)=1, \text{ and} \\ \phi^{-1} \text{ extends onto } \bar{U}_f \text{ as a continuous function} \end{array} \right\}.$$

For a given positive real number α_*, the class

$$\mathcal{IS}[\alpha_*] := \{e^{2\pi\alpha\mathbf{i}}\cdot f \mid f \in \mathcal{IS}, \text{ and } \alpha \in [0,\alpha_*]\}$$

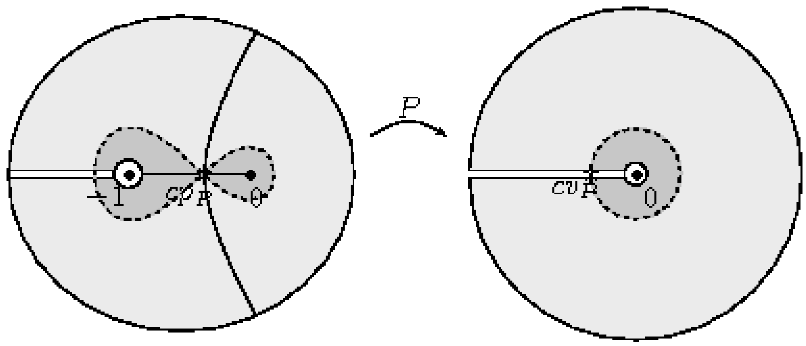

FIGURE 1. A schematic presentation of the Polynomial P, its domain, and its range. Similar colors and linestyles describe the map.

plays a significant role in this study. As the class $\mathcal{IS}[\alpha_*]$ is identified with the space of univalent maps on the unit disk with a neutral fixed point at 0, it is a compact class in the above topology.

Any map $h = e^{2\pi\alpha\mathbf{i}} f_0$ in $\mathcal{IS}[\alpha_*]$, with $\alpha \neq 0$ and $f_0 \in \mathcal{IS}$, has a fixed point at 0 with multiplier $e^{2\pi\alpha\mathbf{i}}$, and another fixed point $\sigma_h \neq 0$. The σ_h fixed point has asymptotic expansion $\sigma_h = -4\pi\alpha\mathbf{i}/f_0''(0) + o(\alpha)$, when h converges to f_0 in a fixed neighborhood of 0. Clearly, $\sigma_h \to 0$ as $\alpha \to 0$.

See Figure 2 for the contents of the following theorem.

Theorem 1.3. *(Inou-Shishikura* [IS06]*) There exist a real number $\alpha_* > 0$, and positive integers $k, \hat{k}$ such that the class $\mathcal{IS}[\alpha_*]$ satisfies the following properties:*

(1) *$h''(0) \neq 0$ for any map $h \in \mathcal{IS}[\alpha_*]$.*
(2) *For any map $h\colon U_h \to \mathbb{C}$ in $\mathcal{IS}[\alpha_*]$, there exist a domain $\mathcal{P}_h \subset U_h$, bounded by piecewise smooth curves, and a univalent map $\Phi_h\colon \mathcal{P}_h \to \mathbb{C}$ with the following properties:*
   (a) *$\mathcal{P}_h$ is compactly contained in U_h. Moreover, it contains the critical point $\mathrm{cp}_h := \varphi(-\frac{1}{3})$ in its interior as well as 0 and σ_h on its boundary.*
   (b) *There exists a continuous branch of argument defined on $\mathcal{P}_h$ such that*
   $$\max_{w,w' \in \mathcal{P}_h} |\arg(w) - \arg(w')| \leq 2\pi\hat{k}$$
   (c) *$\Phi_h(\overline{\mathcal{P}_h}) = \{w \in \mathbb{C} \mid 0 \leq \mathrm{Re}(w) \leq \lfloor 1/\alpha \rfloor - k\}$. Also, $\mathrm{Im}\,\Phi_h(z) \to +\infty$ when $z \in \mathcal{P}_h \to 0$ and $\mathrm{Im}\,\Phi_h(z) \to -\infty$ when $z \in \mathcal{P}_h \to \sigma_h$.*
   (d) *Φ_h satisfies the Abel functional equation, that is,*
   $$\Phi_h(h(z)) = \Phi_h(z) + 1, \text{ whenever } z \text{ and } h(z) \text{ belong to } \mathcal{P}_h.$$
   Moreover, Φ_h is unique once normalized to send cp_h to 0.
   (e) *The map Φ_h depends continuously on h.*

We refer to the univalent map Φ_h obtained in the above theorem as *perturbed Fatou coordinate* or sometimes *Fatou coordinate* for short.

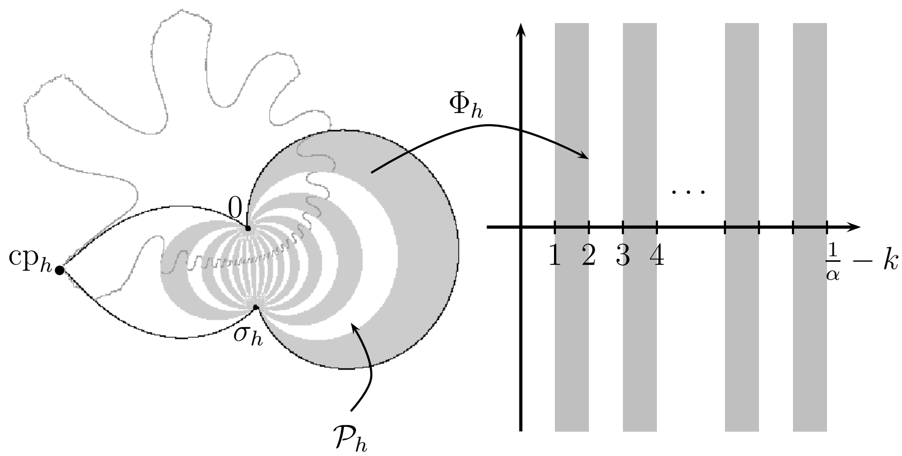

FIGURE 2. A perturbed Fatou coordinate and its domain of definition.

Remark. Parts (b) and (c) in the above theorem (existence of uniform k and $\hat{k}$) are not stated in [IS06] but it follows from their work. These are consequences of the compactness of the class $\mathcal{IS}[\alpha_*]$ and will become clear by Lemma 3.3 which we prove later.

Renormalization: Consider a map $h\colon U_h \to \mathbb{C}$ in $e^{2\pi\alpha\mathbf{i}}\mathcal{IS}$ with $\alpha \le \alpha^*$ (α^* is obtained in Theorem 1.3) and let $\Phi_h\colon \mathcal{P}_h \to \mathbb{C}$ be the normalized Fatou coordinate obtained in that theorem. Define

$$\begin{aligned}&\mathcal{C} := \{z \in \mathcal{P}_h : 1/2 \le \mathrm{Re}(\Phi_h(z)) \le 3/2\,,\ -2 < \mathrm{Im}\,\Phi_h(z) \le 2\}, \text{and}\\ &\mathcal{C}^\sharp := \{z \in \mathcal{P}_h : 1/2 \le \mathrm{Re}(\Phi_h(z)) \le 3/2\,,\ 2 \le \mathrm{Im}\,\Phi_h(z)\}.\end{aligned} \tag{2}$$

By definition, $\mathcal{C}$ contains the critical value of h in its interior, and $\mathcal{C}^\sharp$ contains 0 (fixed point of h) on its boundary. For integers $k > 0$, let $(\mathcal{C}^\sharp)^{-k}$ denote the unique connected component of $h^{-k}(\mathcal{C}^\sharp)$ with 0 on its boundary. Similarly, if there exists a unique connected component of $h^{-k}(\mathcal{C})$ which has non-empty intersection with $(\mathcal{C}^\sharp)^{-k}$, it will be denoted by $\mathcal{C}^{-k}$. Let k_h be the smallest positive integer (if it exists) for which the sets $\mathcal{C}^{-k_h}$ and $(\mathcal{C}^\sharp)^{-k_h}$ are contained in the set

$$\{z \in \mathcal{P}_h \mid 0 < \mathrm{Re}\,\Phi_h(z) < 1/\alpha - k - 1/2\}.$$

For this k_h let

$$S_h := \mathcal{C}^{-k_h} \cup (\mathcal{C}^\sharp)^{-k_h}.$$

Consider the map

$$\Phi_h \circ h^{k_h} \circ \Phi_h^{-1} : \Phi_h(S_h) \to \mathbb{C}. \tag{3}$$

By equivariance property of Φ_h (Abel functional equation), this map projects via $z = \frac{-4}{27}e^{2\pi\mathbf{i}w}$ to a map of the form $z \mapsto e^{2\pi\frac{-1}{\alpha}\mathbf{i}}z + O(z^2)$, defined on some neighborhood of the origin.

Further conjugating this map by $s\colon z \mapsto \bar{z}$, to make the rotation number at 0 positive, we obtain a map $\mathcal{R}(h)$ of the form $z \mapsto e^{\frac{2\pi}{\alpha}\mathbf{i}}z + O(z^2)$. The map $\mathcal{R}(h)$ is called the *near parabolic renormalization* of h by Inou and Shishikura. We simply refer to it as *renormalization* of h. One can see (Lemma 2.1) that one time iterating $\mathcal{R}(h)$ corresponds to several times iterating the map h, or in other words, many times iterating h is equal to composition of two changes of coordinate and one iterate of $\mathcal{R}(h)$. This renormalization is closely related to the Douady-Ghys renormalization introduced in [Dou87], however it is defined on a smaller class of maps but gives a map with a larger domain of definition. For other applications of this theory one may see [Dou94] and [Shi98].

The following theorem in [IS06] states that the above definition of near parabolic renormalization $\mathcal{R}$ can be carried out for maps in $\mathcal{IS}$. For a given positive integer N, let Irr_N denote the set of real numbers $\alpha = [a_0, a_1, a_2, \dots]$ with $a_i \geq N$.

Theorem 1.4. *(Inou-Shishikura) There exist an integer $N > 0$ such that if $h \in e^{2\pi\alpha\mathbf{i}} \cdot \mathcal{IS}$ with $\alpha \in Irr_N$, then $\mathcal{R}(h)$ is well-defined and belongs to the class $\mathcal{IS}[1/N]$ (i.e. it can be written of the form $\mathcal{R}(h) = e^{\frac{2\pi}{\alpha}\boldsymbol{i}} \cdot P \circ \psi^{-1}$).*

The same conclusion holds for the map $P_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z + z^2$, that is, $\mathcal{R}(P_\alpha)$ is well-defined and belongs to $\mathcal{IS}[1/N]$ provided α is small enough and belongs to Irr_N.

Although quadratic polynomials $P_\alpha = e^{2\pi\alpha\mathbf{i}}z + z^2$ do not belong to the class $\mathcal{IS}[\alpha_*]$, the theorem states that one can define the Fatou coordinate for this map and renormalize it by the above definition once α is small enough. Hence, the theorem guaranties that for α in Irr_N, the sequence of renormalizations

$$f_n := \mathcal{R}^n(P_\alpha) : U_n \to \mathbb{C}$$

are defined and belong to the class $\mathcal{IS}[1/N]$. For simplicity of notation, we let $f_0 := P_\alpha$ and $\alpha_0 := \alpha$. Each map f_n has a fixed point at 0 with multiplier $e^{2\pi\alpha_n i}$, $n = 0, 1, 2, \dots$.

2. Accumulation on the fixed point

2.1. **Sectors around the fixed point.** Here we introduce a sequence of subsets of $\mathbb{C}$ containing 0 on their boundary, such that a.e. z in the Julia set of P_α has to visit these sets. From now on we will assume that N is large enough, or $\alpha_* = 1/N$ is small enough, so that the class $\mathcal{IS}[1/N]$ satisfies the conclusions of Theorems 1.3 and 1.4. Moreover, for technical reasons, we will assume that

$$\alpha_* \leq \frac{1}{k + \hat{k}} \tag{4}$$

for k and $\hat{k}$ obtained in Theorem 1.3.

Changes of coordinates: For every $n \geq 0$, let $\Phi_n := \Phi_{f_n}$ denote the Fatou coordinate of the map $f_n\colon U_n \to \mathbb{C}$ defined on the set $\mathcal{P}_n := \mathcal{P}_{f_n}$ introduced in Theorem 1.3. By part (b) of Theorem 1.3 and our assumption (4), there are holomorphic inverse branches $\eta_n\colon \mathcal{P}_n \to \mathbb{C}$ of the map

$$\mathbb{E}\mathrm{xp}(z) := z \longmapsto \frac{-4}{27} s \circ e^{2\pi\mathbf{i}z} : \mathbb{C} \to \mathbb{C}^*, \text{ where } s(z) = \bar{z},$$

with $\eta_n(\mathcal{P}_n) \subset \Phi_{n-1}(\mathcal{P}_{n-1})$. There may be several choices for this map but we choose one of them for each n and fix this choice for the rest of this note.

Now we can define $\psi_n := \Phi_{n-1}^{-1} \circ \eta_n : \mathcal{P}_n \to \mathcal{P}_{n-1}$. Note that each ψ_n extends continuously to $0 \in \partial \mathcal{P}_n$ by mapping it to 0. Consider the maps

$$\Psi_n := \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n : \mathcal{P}_n \to \mathcal{P}_0$$

with values in the dynamical plane of the polynomial f_0.

For every $n = 0, 1, 2, \ldots$, let $\mathcal{C}_n$ and $\mathcal{C}_n^\sharp$ denote the sets obtained in (2) for f_n. Let k_n be the smallest positive integer for which $\mathcal{C}_n^{-k_n}$ and $(\mathcal{C}_n^\sharp)^{-k_n}$ are contained in $\mathcal{P}_n$. We define the sector S_n^0 as

$$S_n^0 := \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n} \subset \mathcal{P}_n.$$

By definition, the critical value of f_n is contained in $f_n^{k_n}(S_n^0)$.

For every $n \geq 0$, define

$$S_n^1 := \psi_{n+1}(S_{n+1}^0) \subset \mathcal{P}_n.$$

In general for $i \geq 2$, let

$$S_n^i := \psi_{n+1} \circ \cdots \circ \psi_{n+i}(S_{n+i}^0) \subset \mathcal{P}_n.$$

All these sectors contain 0 on their boundary. We will mainly work with S_0^i, S_1^i, and S_i^0, for $i = 0, 1, 2, \ldots$. See Figure 3.

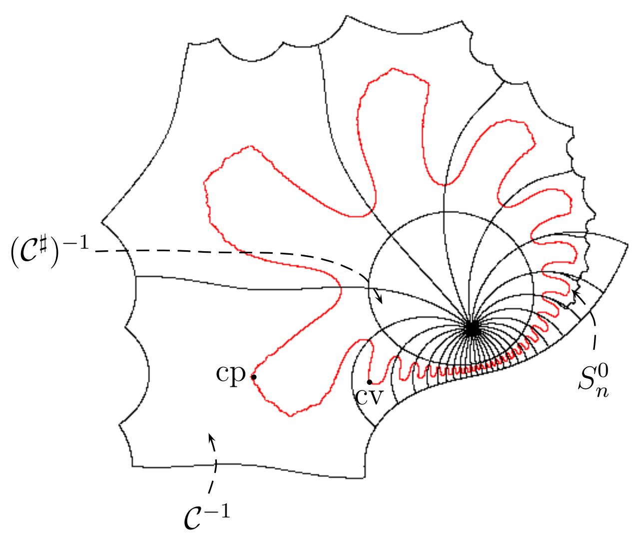

Figure 3. Figure shows the first generation of sectors. The red curve (grey in the printed version) approximates orbit of the critical point.

Lemma 2.1. *Let $z \in \mathcal{P}_{n-1}$ be a point with $w := \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(z) \in \mathrm{Dom}\,(f_n)$. There exists an integer ℓ_z with $2 \leq \ell_z \leq \lfloor \frac{1}{\alpha_{n-1}} \rfloor - k + k_{n-1} + 1$, such that*

- $f_{n-1}^{\ell_z}(z) \in \mathcal{P}_{n-1}$,
- $\mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(f_{n-1}^{\ell_z}(z)) = f_n(w)$,
- $z, f_{n-1}(z), f_{n-1}^2(z), \dots, f_{n-1}^{\ell_z}(z) \in \bigcup_{i=0}^{k_{n-1}+\lfloor \frac{1}{\alpha_{n-1}} \rfloor - k - 1} f_{n-1}^i(S_{n-1}^0)$,

Moreover, if $w \in \operatorname{int} \operatorname{Dom}(f_n)$*, then the orbit*

$$z, f_{n-1}(z), f_{n-1}^2(z), \dots, f_{n-1}^{\ell_z}(z)$$

belongs to the interior of $\bigcup_{i=0}^{k_{n-1}+\lfloor \frac{1}{\alpha_{n-1}} \rfloor - k - 1} f_{n-1}^i(S_{n-1}^0)$

Proof. As $w \in \operatorname{Dom}(f_n)$, by definition of renormalization $\mathcal{R}(f_{n-1}) = f_n$, there are

$$\zeta \in \Phi_{n-1}(S_{n-1}^0), \text{ and } \quad \zeta' \in \Phi_{n-1}(\mathcal{P}_{n-1}), \text{ with } 0 \le \operatorname{Re} \zeta' < 1,$$

such that

$$\mathbb{E}\mathrm{xp}(\zeta) = w, \quad \mathbb{E}\mathrm{xp}(\zeta') = f_n(w), \text{ and } \quad \zeta' = \Phi_{n-1} \circ f_{n-1}^{k_{n-1}} \circ \Phi_{n-1}^{-1}(\zeta).$$

Since $\mathbb{E}\mathrm{xp}(\Phi_{n-1}(z)) = w$ too, and $\zeta \in \Phi_{n-1}(S_{n-1}^0)$, there exists an integer ℓ with $k_{n-1} + 1 \le \ell \le \lfloor \frac{1}{\alpha_{n-1}} \rfloor - k$, such that $\Phi_{n-1}(z) + \ell = \zeta$ (k is the constant obtained in Theorem 1.3).

By equivariance property of Φ_{n-1}, we have

$$\begin{aligned}
\zeta' &= \Phi_{n-1} \circ f_{n-1}^{k_{n-1}} \circ \Phi_{n-1}^{-1}(\zeta) \\
&= \Phi_{n-1} \circ f_{n-1}^{k_{n-1}} \circ \Phi_{n-1}^{-1}(\Phi_{n-1}(z) + l_1) \\
&= \Phi_{n-1} \circ f_{n-1}^{k_{n-1}+\ell} \circ \Phi_{n-1}^{-1}(\Phi_{n-1}(z)) \\
&= \Phi_{n-1} \circ f_{n-1}^{k_{n-1}+\ell}(z).
\end{aligned}$$

If we let $\ell_z := k_{n-1} + \ell + 1$, then we have

$$2 \le \ell_z \le k_{n-1} + \lfloor \frac{1}{\alpha_{n-1}} \rfloor - k + 1, \quad f_{n-1}^{\ell_z}(z) = \Phi_{n-1}^{-1}(\zeta' + 1) \in \mathcal{P}_{n-1},$$

and

$$\mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(f_{n-1}^{\ell_z}(z)) = \mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(\Phi_{n-1}^{-1}(\zeta' + 1)) = \mathbb{E}\mathrm{xp}(\zeta' + 1) = f_n(w).$$

This finishes the first two parts. For the last property, first observe that one of the following two holds

- there exists a positive integer j with $f_{n-1}^j(z) \in S_{n-1}^0$,
- there exists a non-negative integer j with $z \in f_{n-1}^j(S_{n-1}^0)$.

If the first one occurs (this is when ℓ is positive), then

$$z, f_{n-1}(z), \dots, f_{n-1}^{j-1}(z) \in \bigcup_{i=k_{n-1}}^{k_{n-1}+\lfloor \frac{1}{\alpha_{n-1}} \rfloor - k - 1} f_{n-1}^i(S_{n-1}^0),$$

$$\text{and} \quad f_{n-1}^j(z), \dots, f_{n-1}^{\ell_z}(z) \in \bigcup_{i=0}^{k_{n-1}+1} f_{n-1}^i(S_{n-1}^0).$$

If the second one occurs (when ℓ is non-positive), then we have

$$z, f_{n-1}(z), \dots, f_{n-1}^{\ell_z}(z) \in \bigcup_{i=j}^{k_{n-1}+1} f_{n-1}^{i}(S_{n-1}^{0}).$$

The final statement follows from the open mapping property of holomorphic maps. That is, image of every open set under a holomorphic map is open. For example, if $w \in \operatorname{int} \operatorname{Dom}(f_n)$ then $\zeta \in \operatorname{int} \Phi_{n-1}(S_{n-1}^{0})$ which implies that z belongs to the interior of that union. $\square$

In the above lemma there are many choices (indeed $\lfloor \frac{1}{\alpha_{n-1}} \rfloor - k - 1$) for ℓ_z. However, in the following two lemmas we make a specific choice for $f_{n-1}^{\ell_z}(z)$ in order to control ℓ_z.

Lemma 2.2. *For every $n \geq 1$, the two maps*

$$f_n \colon \mathcal{P}_n \to f_n(\mathcal{P}_n) \qquad \text{and} \qquad f_0^{q_n} \colon \Psi_n(\mathcal{P}_n) \to f_0^{q_n}(\Psi_n(\mathcal{P}_n))$$

are conjugate by Ψ_n. That is, the following diagram

$$\begin{tikzcd} \Psi_n(\mathcal{P}_n) \arrow[r, "f_0^{q_n}"] & f_0^{q_n}(\Psi_n(\mathcal{P}_n)) \\ \mathcal{P}_n \arrow[u, "\Psi_n"] \arrow[r, "f_n"] & f_n(\mathcal{P}_n) \arrow[u, "\Psi_n"'] \end{tikzcd}$$

commutes wherever it is defined.

Similarly for every $n > m$, $f_n \colon \mathcal{P}_n \to f_n(\mathcal{P}_n)$ is conjugate to some iterate of f_m defined on the set $\psi_{m+1} \circ \dots \circ \psi_n(\mathcal{P}_n)$.

Proof. By definition of the renormalization $\mathcal{R}$, this property holds near 0 (fixed point). So by analytic continuation, it hold on the domain of definition. The integers q_n are the closest return times for the rotation of angle α_0 around 0. $\square$

The following lemma transfers iterates of the map

$$f_n^{k_n} \colon S_n^0 \to f_n^{k_n}(S_n^0) \subseteq \mathcal{P}_n$$

to the dynamic plane of f_0.

Lemma 2.3. *The map $f_n^{k_n} \colon S_n^0 \to f_n^{k_n}(S_n^0) \subset \mathcal{P}_n$ is conjugate to*

$$f_0^{k_n q_n + q_{n-1}} \colon \Psi_n(S_n^0) \to \Psi_n(f_n^{k_n}(S_n^0)),$$

by Ψ_n.

Proof. Proof of this lemma is similar to the previous one. Here, $k_n q_n + q_{n-1}$ is the return time of the points in $\Psi_n(S_n^0)$ back to $\Psi_n(\mathcal{P}_n)$ under f_0. $\square$

2.2. **Neighborhoods of the post-critical set.** On each level $j \geq 0$, consider the following union:

$$\Omega_j^0 := \bigcup_{i=0}^{k_j+\lfloor\frac{1}{\alpha_j}\rfloor-k-1} f_j^i(S_0^j).$$

Using the two previous lemmas, we transfer these sets to the dynamic plane of f_0 to obtain,

$$\Omega_0^n := \bigcup_{i=0}^{q_{n+1}+(k_n-k-1)q_n} f_0^i(S_0^n),$$

for every $n \geq 0$.

To transfer the sectors in the union Ω_n^0 from level n to the level 0, the first k_n sectors give $k_n q_n + q_{n-1}$ sectors, by Lemma 2.3, and the $\lfloor\frac{1}{\alpha_n}\rfloor - k - 1$ remaining ones produce $q_n(\lfloor\frac{1}{\alpha_n}\rfloor - k - 1)$ number of sectors, by Lemma 2.2. Thus, totally we obtain

$$\begin{aligned}(k_n q_n + q_{n-1}) + q_n(\lfloor\frac{1}{\alpha_n}\rfloor - k - 1) &= q_n(\lfloor\frac{1}{\alpha_j}\rfloor + q_{n-1}) + q_n(k_n - k - 1)\\ &\leq q_{n+1} + q_n(k_n - k - 1),\end{aligned}$$

number of sectors, by formula $q_{n+1} = a_{n+1}q_n + q_{n-1}$.

Proposition 2.4. *For every $n \geq 0$, we have the following:*

(1) *Ω_0^{n+1} is compactly contained in the interior of Ω_0^n,*
(2) *The post-critical set of f_0 is contained in the interior of Ω_0^n.*

Proof.
Part (1): To show that $\Omega_0^{n+1} \subset \Omega_0^n$, it is enough to show that for every $z \in S_0^{n+1}$ there are points $z_1, z_2, \ldots, z_m$ in S_0^n as well as non-negative integers $t_1, t_2, \ldots, t_{m+1}$, for some positive integer m (indeed $m = k_{n+1} + \lfloor\frac{1}{\alpha_{n+1}}\rfloor - k - 1$), with the following properties:

- $f_0^{t_1}(z_1) = z$, and $f_0^{t_{m+1}}(z_m) = f_0^{q_{n+2}+(k_{n+1}-k-1)q_{n+1}}(z)$,
- $f_0^{t_j}(z_{j-1}) = z_j$, for $j = 2, 3, \ldots, m$,
- $t_j \leq q_{n+1} + (k_n - k - 1)q_n$, for every $j = 1, 2, \ldots, m + 1$.

For $z \in S_0^{n+1}$, let $\zeta := \Psi_{n+1}^{-1}(z) \in S_{n+1}^0$. By definition of S_{n+1}^0, the iterates

$$\zeta, f_{n+1}(\zeta), f_{n+1}^2(\zeta), \ldots, f_{n+1}^{k_{n+1}+\lfloor\frac{1}{\alpha_{n+1}}\rfloor-k-1}(\zeta)$$

are defined and belong to Dom f_{n+1}.

By Lemma 2.1 applied to $\psi_{n+1}(\zeta)$, there are two points ξ_1 and ξ_2 in the set $\psi_n(\mathcal{P}_n)$ (take $\xi_1 = \psi_{n+1}(\zeta)$) as well as a positive integer ℓ_0 with

$$\mathbb{E}\mathrm{xp} \circ \Phi_n(\xi_1) = \zeta, \;\; \mathbb{E}\mathrm{xp} \circ \Phi_n(\xi_2) = f_{n+1}(\zeta), \text{ and } f_n^{\ell_0}(\xi_1) = \xi_2.$$

Let $\sigma_1 \in S_n^0$ and an integer ℓ_1, with $1 \leq \ell_1 \leq k_n + a_{n+1} - k - 1$, be such that $f_n^{\ell_1}(\sigma_1) = \xi_1$. By the same lemma, there is a point σ_2 in the orbit

$$\xi_1, f_n(\xi_1), f_n^2(\xi_1), \ldots, f_n^{\ell_0-1}(\xi_1), \xi_2$$

which belongs to S_n^0. Let ℓ_2 denote the positive integer with $1 \leq \ell_2 \leq k_n + \lfloor \frac{1}{\alpha_n} \rfloor - k - 1$ and $f_n^{\ell_2}(\sigma_1) = \sigma_2$.

By the same argument for ξ_2, which satisfies $\mathbb{E}\mathrm{xp} \circ \Phi_n(\xi_2) = f_{n+1}(\zeta)$, we obtain $\sigma_3 \in S_n^0$, $\xi_3 \in \mathcal{P}_n$ and a positive integer ℓ_3 with $1 \leq \ell_3 \leq k_n + a_{n+1} - k - 1$, such that $f_n^{\ell_3}(\sigma_2) = \sigma_3$.

Repeating this argument with $\xi_3, \xi_4, \ldots, \xi_m$, for $m = k_{n+1} + \lfloor \frac{1}{\alpha_{n+1}} \rfloor - k - 1$, one obtains a sequence

$$\sigma_1, \sigma_2, \ldots, \sigma_m$$

of points in S_n^0 as well as positive integers

$$\ell_1, \ell_2, \ldots, \ell_{k_{n+1} + \lfloor \frac{1}{\alpha_{n+1}} \rfloor - k - 1}, \ell_{m+1},$$

all bounded by $k_n + \lfloor \frac{1}{\alpha_n} \rfloor - k - 1$, which satisfy:

- $f_n^{\ell_{j+1}}(\sigma_j) = \sigma_{j+1}$, for all $j = 2, 3, \ldots, m-1$,
- $f_n^{\ell_1}(\sigma_1) = \xi_1$ and $f_n^{\ell_{m+1}}(\sigma_m) = \xi_m$.

Now we define $z_i := \Psi_n(\sigma_i) \in S_0^n$, for $j = 1, 2, \ldots, m$. By definition $\Psi_n(\xi_1) = z$.

We claim that $\Psi_n(\xi_m) = f_0^{q_{n+2} + (k_{n+1} - k - 1)q_{n+1}}(z)$. This is because

$$\mathbb{E}\mathrm{xp} \circ \Phi_n(\xi_m) = f_{n+1}^{k_{n+1} + \lfloor \frac{1}{\alpha_{n+1}} \rfloor - k - 1}(\zeta)$$

which is mapped to $f_0^{q_{n+2} + (k_{n+1} - k - 1)q_{n+1}}(z)$ under Ψ_{n+1}, using Lemmas 2.2 and 2.3 with f_0 and f_{n+1}.

By Lemmas 2.2 and 2.3, ℓ_j times iterating f_n corresponds to t_j times iterating f_0, for each $j = 1, 2, \ldots, \ell_{m+1}$. By the same lemmas, as ℓ_j is bounded by $k_n + \lfloor \frac{1}{\alpha_n} \rfloor - k - 1$, each t_j is bounded by

$$k_n q_n + q_{n-1} + (\lfloor \frac{1}{\alpha_n} \rfloor - k - 1) q_n \leq q_{n+1} + q_n(k_n - k - 1).$$

To show that Ω_0^{n+1} is compactly contained in the interior of Ω_0^n, we use the open mapping property of holomorphic maps. That is, if z' is a point in the closure of Ω_0^{n+1}, there exists a point z in the closure of S_0^{n+1} with $f_0^{t_0}(z) = z'$, for a non-negative integer t_0 less than $q_{n+2} + (k_n - k - 1)q_{n+1}$. The last statement in Lemma 2.1 implies that all the points σ_j in the above argument can be chosen in the interior of S_n^0. Hence, every z_i is contained in the interior of S_0^n.

Part (2): First we claim that for every $n \geq 0$, the critical point of f_0 belongs to Ω_0^n, and in addition, it can be iterated at least $(a_{n+1} - k - 1)q_n$ times within this set.

To prove the claim, note that $f_n \colon S_n^0 \to f_n^{k_n}(S_n^0)$ has a critical point. Thus, by Lemma 2.3, $f_0^{k_n q_n + q_{n+1}} \colon S_0^n \to \Psi_n(f_n^{k_n}(S_n^0))$ also has a critical point. That means that the critical point of f_0 is contained in the union $\cup_{i=0}^{k_n q_n + q_{n-1}} f_0^i(S_0^n)$. Therefore, by definition of Ω_0^n, the critical point can be iterated at least

$$q_{n+1} + (k_n - k - 1)q_n - k_n q_n - q_{n-1} = (a_{n+1} - k - 1)q_n$$

times with in Ω_0^n.

As $a_{n+1} - k - 1 \geq 1$, and q_n growth (exponentially) to infinity as n goes to infinity, part (1) implies that the critical point of f_0 can be iterated infinite number of times within each Ω_0^n.

For every $n \geq 0$, Ω_0^n contains closure of Ω_0^{n+1} in its interior and Ω_0^{n+1} contains orbit of the critical point. Therefore, the post-critical set is contained in int Ω_0^n. □

In the next lemma we show that all the sectors contained in the union Ω_0^n are visited by almost every point in the Julia set of f_0. That is,

Lemma 2.5. *Let n and ℓ be positive integers with $0 \leq \ell \leq q_{n+1} + (k_n - k - 1)q_n$. Then for almost every z in the Julia set of f_0, there exists a non-negative integer ℓ_z with $f_0^{\ell_z}(z) \in f_0^{\ell}(S_0^n)$.*

Proof. It is enough to prove the lemma for $\ell = 0$. We claim that for every $n \geq 0$, the set of points which visit Ω_0^{n+1} contains the set of points which visit S_0^n. Assuming the claim for a moment, because the set of points in the Julia set that visit Ω_0^{n+1} has full measure by Propositions 1.2 and 2.4, we can conclude the lemma.

To prove the claim, let z be an arbitrary point in J for which there exists an integer $t_1 \geq 0$ with $f_0^{t_1}(z) \in \Omega_0^{n+1}$. Let $t_2 \geq t_1$ be a positive integer with

$$f_0^{t_2}(z) \in f_0^{q_{n+2}+(k_{n+1}-k-1)q_{n+1}}(S_0^{n+1}).$$

Let ξ denote the point $\Psi_n^{-1}(f_0^{t_2}(z))$ in $\mathcal{P}_n$. As $\zeta := \Psi_{n+1}^{-1}(f_0^{t_2}(z)) = \mathbb{E}\mathrm{xp} \circ \Phi_n(\xi)$ belongs to $\mathcal{P}_{n+1}$, $f_{n+1}(\zeta)$ is defined. Hence, by Lemma 2.1, there exists a non-negative integer j such that the orbit $\xi, f_n(\xi), f_n^2(\xi), \ldots, f_n^j(\xi)$ is contained in $\mathcal{P}_n$ and the last point $f_n^j(\xi)$ is contained in S_n^0. By Lemma 2.2, $f_0^{t_2+q_nj}(z)$ belongs to S_0^n. □

2.3. **Size of the sectors.** Now we want to control size of certain sectors contained in the unions Ω_0^n in terms of the Brjuno function. The following two lemmas are our main technical tools. Their proof comes at the end of this section.

Lemma 2.6. *There exists a constant $M \geq 1$ such that for every integer $n \geq 1$ there exists an integer $\tau(n)$ with $k_n \leq \tau(n) \leq a_{n+1} - k - 2$, and*

$$\operatorname{diam}(f_n^{\tau(n)}(S_n^0)) \leq M\alpha_n.$$

Lemma 2.7. *There exists a constant $M \geq 1$ such that for every integer $n \geq 1$, there exists an integer $\kappa(n)$, with $0 \leq \kappa(n) \leq a_{n+1} - k - 1$ that satisfies the following:*

For every $w \in \mathcal{P}_{n+1}$,

1. $f_n^{\kappa(n)} \circ \psi_{n+1}(w) \in \mathcal{P}_n$,
2. $|f_n^{\kappa(n)} \circ \psi_{n+1}(w)| \leq M\alpha_n |w|^{\alpha_n}$.

From now on we assume that M denotes a constant which satisfies these two lemmas.

Proposition 2.8. *There exists a constant C such that for every $m \geq 1$, there exist non-negative integers $\gamma(m)$ and $\gamma'(m) \leq q_{n+1} + (k_n - k - 1)q_n$ for which the following holds*

1. $\operatorname{diam}(f_1^{\gamma(m)}(S_1^m)) \leq C \cdot \alpha_1 \cdot \alpha_2^{\alpha_1} \cdot \alpha_3^{\alpha_1\alpha_2} \cdot \alpha_4^{\alpha_1\alpha_2\alpha_3} \ldots \alpha_m^{\alpha_1\ldots\alpha_{m-1}}$.
2. $f_1^{\gamma(m)}(S_1^m) \subseteq \mathcal{P}_1$

(3) $\psi_1(f_1^{\gamma(m)}(S_1^m)) = f_0^{\gamma'(m)}(S_0^{m+1})$, *that is,* $\psi_1(f_1^{\gamma(m)}(S_1^m))$ *is among the sectors in the union* Ω_0^{m+1}.

Proof. For the constant M obtained for the two Lemmas 2.6 and 2.7, let

$$\begin{aligned} C &= M \cdot M^{\alpha_1} \cdot M^{\alpha_1\alpha_2} \cdot M^{\alpha_1\alpha_2\alpha_3} \dots \\ &= M^{1+\alpha_1+\alpha_1\alpha_2+\alpha_1\alpha_2\alpha_3+\dots} \\ &\le M^{1+1/2+1/2^2+1/2^3+\dots} = M^2 < \infty. \qquad (\text{as } \alpha_i < 1/2) \end{aligned}$$

Given $m \ge 1$, by Lemma 2.6, there exists $\tau(m)$ with $k_m \le \tau(m) \le k_m - k - 2$, and

$$\operatorname{diam}(f_m^{\tau(m)}(S_m^0)) \le M \cdot \alpha_m.$$

By Lemma 2.7 with $n = m-1$, and $w \in f_m^{\tau(m)}(S_m^0)$, we obtain

$$\operatorname{diam}(f_{m-1}^{\kappa(m-1)} \circ \psi_m(f_m^{\tau(m)}(S_m^0))) \le M \cdot \alpha_{m-1} \cdot (M \cdot \alpha_m)^{\alpha_{m-1}}.$$

Now by Lemma 2.2 we have

$$\operatorname{diam}(f_{m-1}^{\kappa(m-1)} \circ f_{m-1}^{\tau(m)a_{m+1}+1}(\psi_m(S_m^0)) \le M \cdot M^{\alpha_{m-1}} \cdot \alpha_{m-1} \cdot \alpha_m^{\alpha_{m-1}}$$

or equivalently

$$\operatorname{diam}(f_{m-1}^{\kappa(m-1)+\tau(m)a_{m+1}+1}(S_{m-1}^1) \le M \cdot M^{\alpha_{m-1}} \cdot \alpha_{m-1} \cdot \alpha_m^{\alpha_{m-1}}.$$

Again applying Lemma 2.7 with $n = m-2$, the last inequality implies that

$$\begin{aligned} &\operatorname{diam}(f_{m-2}^{\kappa(m-2)} \circ \psi_{m-1}(f_{m-1}^{\kappa(m-1)+\tau(m)a_{m+1}+1}(S_{m-1}^1)) \le \\ &\qquad\qquad M \cdot \alpha_{m-2} \cdot (M \cdot M^{\alpha_{m-1}} \cdot \alpha_{m-1} \cdot \alpha_m^{\alpha_{m-1}})^{\alpha_{m-2}}. \end{aligned}$$

which is equivalent, by Lemma 2.2, to

$$\begin{aligned} &\operatorname{diam}(f_{m-2}^{\kappa(m-2)+(\kappa(m-1)+\tau(m)a_{m+1}+1)a_m+1}(\psi_{m-1}(S_{m-1}^1))) \le \\ &\qquad\qquad M \cdot \alpha_{m-2} \cdot (M \cdot M^{\alpha_{m-1}} \cdot \alpha_{m-1} \cdot \alpha_m^{\alpha_{m-1}})^{\alpha_{m-2}}. \end{aligned}$$

Repeatedly using Lemma 2.7 with $n = m-3, m-4, \dots, 1$, one obtains

$$\operatorname{diam}(f_1^{\gamma(m)}(S_1^m)) \le M \cdot \alpha_1 \cdot [M \cdot \alpha_2[M \cdot \alpha_3[\dots[M \cdot \alpha_m]^{\alpha_{m-1}}]^{\alpha_{m-2}} \dots]^{\alpha_2}]^{\alpha_1}$$

for some integer $\gamma(m)$. Therefore $f_1^{\gamma(m)}(S_1^m)$ has diameter less than

$$\begin{aligned} &M \cdot M^{\alpha_1} \cdot M^{\alpha_1\alpha_2} \dots M^{\alpha_1\alpha_2\dots\alpha_{m-1}} \cdot \alpha_1 \cdot \alpha_2^{\alpha_1} \cdot \alpha_3^{\alpha_1\alpha_2} \cdot \alpha_4^{\alpha_1\alpha_2\alpha_3} \dots \alpha_m^{\alpha_1\dots\alpha_{m-1}} \le \\ &\qquad\qquad C \cdot \alpha_1 \cdot \alpha_2^{\alpha_1} \cdot \alpha_3^{\alpha_1\alpha_2} \cdot \alpha_4^{\alpha_1\alpha_2\alpha_3} \dots \alpha_m^{\alpha_1\dots\alpha_{m-1}}. \end{aligned}$$

this finishes the first Part of the proposition.

The second part of the proposition follows from above argument when Lemma 2.7 was used with $m = 1$.

To see the third statement in the proposition, first note that $\tau(m)$ is chosen strictly less than $a_{m+1} + k_m - k - 1$. Therefor, $\psi_1(f_1^{\gamma(m)}(S_1^m))$ is among the sectors in the union Ω_0^{m+1}. Indeed, one can see that $\gamma'(m)$ is strictly between $k_m q_m + q_{m-1}$ and $q_{m+1} + (k_m - k - 1)q_m$. $\square$

Lemma 2.9. *The sequence*

$$\{\alpha_1\alpha_2^{\alpha_1}\alpha_3^{\alpha_1\alpha_2}\alpha_4^{\alpha_1\alpha_2\alpha_3}\dots\alpha_k^{\alpha_1\dots\alpha_{k-1}}\}_k$$

converges to zero as $k\to\infty$, if and only if the Brjuno sum

$$\sum_{n=0}^{\infty}\frac{\log q_{n+1}}{q_n}$$

is divergent.

This is a purely combinatorial lemma whose proof will come in the Appendix.

Proof of Theorem A. The set of points in the Julia set of f_0 which accumulate at the 0 fixed point is equal to the intersection of the sets

$$A_n=\{z\in J:\mathcal{O}(z)\cap B(0,1/n)\neq\varnothing\}.$$

for $n=1,2,\dots$. To prove the theorem, it is enough to show that every A_n has full Lebesgue measure in the Julia set. As the map $\psi_1\colon\mathcal{P}_1\to\mathcal{P}_0$ has continuous extension to the boundary point 0, there exists as $\delta_n>0$ such that if $|w|<\delta_n$, for some $w\in\mathcal{P}_1$, then $|\psi_1(w)|<1/n$.

By Lemma 2.9, there exists an integer $m>0$, for which

$$C\cdot\alpha_1\alpha_2^{\alpha_1}\alpha_3^{\alpha_1\alpha_2}\alpha_4^{\alpha_1\alpha_2\alpha_3}\dots\alpha_k^{\alpha_1\dots\alpha_{k-1}}$$

is less than δ_n, where C is the constant obtained in Proposition 2.8. Now by part (1) and (2) of Proposition 2.8, $\psi_1(f_1^{\gamma(m)}(S_1^m))$ is contained in $B(0,1/n)$. Part (3) of Proposition 2.8 and Lemma 2.5 implies that this set is visited by almost every point in the Julia set of f_0. This completes our proof of Theorem A. □

A corollary of our proof of Theorem A is the following.

Corollary 2.10. *Let $P(z)=z(1+z)^2$, U be the domain defined in* (1)*, and α be a non-Brjuno number in Irr_N. If h is a rational map of the Riemann sphere with the following properties*

- *$h(0)=0$, $h'(0)=1$,*
- *$h(c)\notin U$, if c is a critical point of h.*

Then the rational map $g(z):=e^{2\pi\alpha\mathbf{i}}\cdot P\circ h$ is non-linearizable at 0.

Note that we do not assume in the above corollary that the corresponding Julia set has positive area.

Proof. Above conditions on h implies that g restricted to U belongs to $\mathcal{IS}[\alpha_*]$. All the sectors S_0^n are defined for g and satisfy the estimates in Proposition 2.8. Thus, the critical point which visits all the sectors, by definition, must accumulate at the fixed point. Hence, g is not linearizable at 0. □

3. Perturbed Fatou coordinate

3.1. **Unwrapping the coordinate.** In order to prove Lemmas 2.6 and 2.7 in this section, we will give an approximate formula for the Fatou coordinate Φ_h with a bound on its error.

Assume $h(z) = e^{2\pi\alpha\mathbf{i}} \cdot P \circ \varphi^{-1}(z) \colon \varphi(U) \to \mathbb{C}$ belongs to the class $e^{2\pi\alpha\mathbf{i}}\mathcal{IS}$, and σ_h denotes its non-zero fixed point. In [IS06], N was chosen large enough so that $h(z)$ has only two fixed points 0 and σ_h in its domain of definition. Therefore, one can write $h(z)$ as

$$h(z) = z + z(z - \sigma_h)u_h(z)$$

where $u_h(z)$ is a non-zero holomorphic function defined on the set $\varphi(U)$. Differentiating both sides of this equation at 0, one obtains

$$\sigma_h = \frac{1 - e^{2\pi\alpha\mathbf{i}}}{u_h(0)}. \tag{5}$$

Note also that the map $u_h(z) = (h(z) - z)/(z(z - \sigma_h))$ depends continuously on the map $h(z)$.

Let

$$\tau_h(w) := \frac{\sigma_h}{1 - e^{-2\pi\mathbf{i}\alpha w}}$$

be the universal covering of the Riemann sphere minus two points 0 and σ_h that has deck transformation group generated by

$$T_\alpha(w) := w + \frac{1}{\alpha}.$$

One can see that $\tau_h(w) \to 0$, as $\operatorname{Im}(\alpha w) \to \infty$, and $\tau_h(w) \to \sigma_h$, as $\operatorname{Im}(\alpha w) \to -\infty$.

Define the map

$$F_h(w) := w + \frac{1}{2\pi\alpha\mathbf{i}} \log\big(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)}\big), \qquad \text{with } z = \tau_\alpha(w)$$

on the set of points w with $\tau_h(w) \in \operatorname{Dom}(h)$. The branch of log in the above formula is determined by $-\pi < \operatorname{Im}\log(\cdot) < \pi$. The map F_h is defined on the inverse image of $\operatorname{Dom}(h)$ under τ_α.

It is immediate calculation to see that

$$h \circ \tau_h = \tau_h \circ F_h, \quad \text{and} \quad T_\alpha \circ F_h = F_h \circ T_\alpha.$$

Indeed, F_h was defined using this relations.

We will see in a moment that the Fatou coordinate of a map in the class $\mathcal{IS}[\alpha_*]$ is "essentially" equal to τ_α for our purposes. Hence, we wish to further control τ_h on certain domains.

For every real number $R > 0$, let $\Theta(R)$ denote the set

$$\Theta(R) := \mathbb{C} \setminus \bigcup_{n\in\mathbb{Z}} T_\alpha^n(B(0, R)).$$

Lemma 3.1. *There exists a positive constant C_1 such that*

(1) *For every $Y>0$, there exists $\varepsilon_Y>0$, such that for every $h\in e^{2\pi\alpha\mathbf{i}}\mathcal{IS}$, with $\alpha<\varepsilon_Y$, we have*

$$\forall w\in\Theta(Y),\quad |\tau_h(w)|\leq C_1/Y$$

(2) *For every $r\in(0,1/2)$, $w\in\Theta(\frac{r}{\alpha})$, and every $h\in\mathcal{IS}[\alpha_*]$ we have*

$$|\tau_h(w)|\leq C_1\frac{\alpha}{r}e^{-2\pi\alpha\operatorname{Im}w}.$$

Proof. The σ_h fixed point has the form (5) in terms of u_h, where u_h is a non-zero function in a compact class. Therefore, there exists a constant C' such that for every $h\in e^{2\pi\alpha\mathbf{i}}\mathcal{IS}$, $|\sigma_h|<C'\alpha$. The rest follows from analyzing the explicit formula $1/(1-e^{-2\pi\mathbf{i}\alpha w})$ on $\Theta(Y)$.

To see Part (1), fix an arbitrary $Y>0$. If $w\in\Theta(Y)$, then $1-e^{2\pi\mathbf{i}\alpha w}$ belongs to complement of the ball of radius $e^{-2\pi\alpha Y}-1$ centered at 0 in $\mathbb{C}$. Therefore

$$|\frac{\sigma_h}{1-e^{-2\pi\alpha Y}}|<\frac{C'\alpha}{3\pi\alpha Y}=\frac{C'}{3\pi Y}$$

for small α.

To see part (2), one can observe that for such a w, $|1-e^{-2\pi\mathbf{i}\alpha w}|\geq e^{2\pi r}-1$, and conclude that there exists a constant C'' with

$$|1-e^{-2\pi\mathbf{i}\alpha w}|\geq C''re^{2\pi\alpha\operatorname{Im}w}.$$

This implies that

$$|\tau_h(w)|\leq\frac{C'}{C''}\frac{\alpha}{r}e^{-2\pi\alpha\operatorname{Im}w}.$$

Now we can take C_1 as the maximum of $\frac{C'}{3\pi}$ and $\frac{C'}{C''}$. ☐

3.2. **Estimating the lifted map.**

Lemma 3.2. *There exist positive constants $\varepsilon_0>0$, C_2, C_3, C_4 as well as a positive integer j_0, such that for every map $h\in e^{2\pi\alpha\mathbf{i}}\mathcal{IS}$, with $\alpha\leq\varepsilon_0$, the induced map F_h is defined and is univalent on $\Theta(C_2)$, and moreover*

(1) *For all $w\in\Theta(C_2)$, we have*

$$|F_h(w)-(w+1)|<1/4,\quad \textit{and}\quad |F_h'(w)-1|<1/4.$$

(2) *For every $r\in(0,1/2)$, and $w\in\Theta(\frac{r}{\alpha}+1)$, we have*

$$|F_h(w)-(w+1)|<C_3\frac{\alpha}{r}e^{-2\pi\alpha\operatorname{Im}w},\quad \textit{and}\quad |F_h'(w)-1|<C_3\frac{\alpha}{r}e^{-2\pi\alpha\operatorname{Im}w}.$$

(3) $\mathrm{cp}_h\in B(0,2)\setminus B(0,.22)$. *If $i(h)$ is the smallest non-negative integer with $\operatorname{Re}F_h^{i(h)}(\mathrm{cp}_{F_h})\geq C_2$, then $i(h)\leq\min\{j_0,1/\alpha\}$.*

(4) *For every positive integer $j\leq j_0+\frac{2}{3\alpha}$, we have*

$$|\operatorname{Im}F_h^j(\mathrm{cp}_{F_h})|\leq C_4(1+\log j),\quad \textit{and}\quad |\operatorname{Re}F_h^j(\mathrm{cp}_{F_h})-j|\leq C_4(1+\log j).$$

Proof.

Parts (1) *and* (2): Consider a map $h = e^{2\pi\alpha\mathbf{i}} P \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{C}$ in $e^{2\pi\alpha\mathbf{i}}\mathcal{IS}$. As $\mathrm{cp}_P = -1/3 \notin B(0, 1/3)$, by Koebe Distortion Theorem, $\mathrm{cp}_h = \phi(-1/3) \notin B(0, 1/12)$. So, every h in the above class is defined and univalent on $B(0, 1/12)$. Applying part (1) of Lemma 3.1 with $Y = 12C_1$, we obtain an $\varepsilon_0 > 0$ such that if $\alpha \leq \varepsilon_0$, then

$$\tau_h(\Theta(Y)) \subset B(0, 1/12).$$

Therefore, the induced map F_h is defined and univalent on $\Theta(Y)$.

For $w \in \Theta(Y)$, using notation $\lambda = e^{2\pi\alpha\mathbf{i}}$, we have

$$\begin{aligned} F_h(w) &= w + \frac{1}{2\pi\alpha\mathbf{i}} \log\Big(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)}\Big), \quad \text{with} \quad z = \tau_h(w) \\ &= w + 1 + \frac{1}{2\pi\alpha\mathbf{i}} \log\Big(\frac{1}{\lambda}(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)})\Big). \end{aligned}$$

Now assume we want to show that

$$|F_h(w) - (w+1)| = \Big|\frac{1}{2\pi\alpha} \log\Big(\frac{1}{\lambda}(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)})\Big)\Big| < A$$

for some A with $0 < A < 1/4$. As $2\pi\alpha A < 1$, it is enough to prove

$$\frac{1}{2\pi\alpha}\Big|\frac{1}{\lambda}(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)}) - 1\Big| < \frac{A}{e}.$$

As $|\lambda| = 1$, it is enough to show that

$$\frac{1}{2\pi\alpha}\Big|1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)} - \lambda\Big| < \frac{A}{e}.$$

Replacing σ_h by its value from equation (5), and using $|1 - \lambda| < 2\pi\alpha$, we obtain

$$\frac{1}{2\pi\alpha}\Big|(1-\lambda)(1 - \frac{u_h(z)}{(1 + z u_h(z)) u_h(0)})\Big| < \Big|1 - \frac{u_h(z)}{(1 + z u_h(z)) u_h(0)}\Big|.$$

Since u_h belongs to a compact class, it is possible to make

$$1 - \frac{u_h(z)}{(1 + z u_h(z)) u_h(0)} \tag{6}$$

less than $\frac{1}{4e}$ by restricting z to a sufficiently small disk of radius δ around 0. So,

$$\Big|1 - \frac{u_h(z)}{(1 + z u_h(z)) u_h(0)}\Big| < \frac{1}{4e} = \frac{1/4}{e}, \text{ on } B(0, \delta).$$

By Lemma 3.1, part (1), there is a constant $C'(\delta) \geq Y$ such that $|z| = |\tau_\alpha(w)| < \delta$ holds for every $w \in \Theta(C'(\delta))$. With this constant $C'(\delta)$ (for C_2), we have the first inequality in part (1) of the lemma.

The second inequality in part (1) follows from the Cauchy estimate (integral formula) applied to $F_h(w) - w - 1$, once we restrict w to smaller domain $\Theta(C'(\delta) + 1)$. Hence, for $C_2 := C'(\delta) + 1$ we have both inequalities.

For the first inequality in Part (2), using Taylor's Theorem for Expression (6), one obtains

$$1-\frac{u_h(z)}{(1+zu_h(z))u_h(0)}<2\cdot|\frac{u_h'(0)}{u_h(0)-1}||z|.$$

Moreover, as u_h belongs to a compact class,

$$|\frac{u_h'(0)}{u_h(0)-1}|<D'$$

for some constant D'.

Using part (2) of Lemma 3.1, we conclude that for every $w\in\Theta(\frac{r}{\alpha})$, we have

$$\begin{aligned}|1-\frac{u_h(z)}{(1+zu_h(z))u_h(0)}|&<2D'C_1\frac{\alpha}{r}e^{-2\pi\alpha\operatorname{Im}w}\\&=\frac{\frac{2D'C_1}{e}\frac{\alpha}{r}e^{-2\pi\alpha\operatorname{Im}w}}{e}.\end{aligned}$$

This proves the first inequality by introducing $C_3:=2D'C_1/e$.

The other inequality in (2) is also a consequence of Cauchy estimate, once we restrict w to $\Theta(\frac{r}{\alpha}+1)$.

Part (3) : By explicit calculation one can see that $e^{-2\pi\alpha}h$ is univalent on the ball $B(0,1-\sqrt{8/27}e^{-2\pi})\supset B(0,2/3)$. Koebe distortion Theorem applied to this map on $B(0,2/3)$ implies that $\mathrm{cp}_h\in B(0,2)\setminus B(0,.22)$.

By above argument, there is a choice of cp_{F_h} in $\tau_\alpha^{-1}(\mathrm{cp}_h)$ that belongs to a compact subset of $\mathbb{C}$ (independent of α). Since h converges to maps in the compact class $\mathcal{IS}$ as $\alpha\to 0$, cp_{F_h} visits $\Theta(C_2)$ in a finite number of iterates $i(h)$, uniformly bounded by some constant j_0 independent of h. For the same reason,

$$|F_h^i(\mathrm{cp}_{F_h})|\le C',\ \text{for } i=0,1,\dots i(h)$$

for some constant C'.

Part (4) :

It is enough to prove the inequalities for small values of α. For larger α, there are only finite number of iterates to consider. Therefore, by our previous argument in part (3), the inequalities hold for large enough C_4. So, in the following we assume that $\alpha\le\min\{\frac{C_2}{3}+\frac{5j_0+5}{24},\frac{1}{8C_2+41}\}$.

By the first part of this lemma, at each step j with $i(h)\le j\le j_0+\frac{2}{3\alpha}$, we have $F_h^j(\mathrm{cp}_{F_h})\in\Theta(C_2)$,

$$C'+\frac{3j}{4}\le\operatorname{Re}F_h^j(\mathrm{cp}_{F_h})\le C'+\frac{5}{4}+\frac{5j}{4}.$$

and,

$$-C'-\frac{1}{6\alpha}\le -C'-\frac{j}{4}\le\operatorname{Im}F_h^j(\mathrm{cp}_{F_h})\le C'+\frac{j}{4}\le C'+\frac{1}{6\alpha}.$$

Now, one can use part (2) with $r_j = \frac{j\alpha}{6}$ at $F_h^j(\mathrm{cp}_{F_h})$, for $j = i(h), \dots, j_0 + \frac{2}{3\alpha}$, to obtain:

$$\begin{aligned}|F_h(F_h^j(\mathrm{cp}_{F_h})) - F_h^j(\mathrm{cp}_{F_h}) - 1| &\le C_3 \frac{6}{j} e^{2\pi\alpha(C' + \frac{1}{6\alpha})}\\ &\le 6C_3 e^{\pi(C'+\frac{1}{3})}\frac{1}{j}\end{aligned}$$

Putting above inequalities together using triangle inequality, we obtain the following estimates for every $j \le j_0 + \frac{2}{3\alpha}$:

$$\begin{aligned}|\operatorname{Im} F_h^j(\mathrm{cp}_{F_h})| &\le C' + 6C_3 e^{\pi(C'+\frac{1}{3})} \sum_{m=i(h)}^{j} \frac{1}{m}\\ &\le C' + 6C_3 e^{\pi(C'+\frac{1}{3})}(1 + \log j)\\ &\le C' + 6C_3 e^{\pi(C'+\frac{1}{3})}(1 + \log \frac{2}{3\alpha}),\end{aligned}$$

similarly,

$$\begin{aligned}\operatorname{Re} F_h^j(\mathrm{cp}_{F_h}) &\le C' + (j - i(h)) + 6C_3 e^{\pi(C'+\frac{1}{3})}(1 + \log j), \text{ and}\\ \operatorname{Re} F_h^j(\mathrm{cp}_{F_h}) &\ge -C' + (j - i(h)) - 6C_3 e^{\pi(C'+\frac{1}{3})}(1 + \log |j - j_0|)\end{aligned}$$

This finishes part (4) by introducing the appropriate constant C_4. □

3.3. **Estimating the linearizing coordinate.** The following lemma is repeating the existence of the Fatou coordinate stated in the second part of Theorem 1.3. There is a standard argument based on the measurable Riemann mapping theorem to build such a coordinate. Since we need to further analyze it, we repeat this argument in the next lemma. It is mainly following [Shi00].

For real constant $Q > 0$, let Σ_Q denote the set

$$\begin{aligned}\Sigma_Q := \{w \in \mathbb{C} : Q \le \operatorname{Re}(w) \le \frac{1}{\alpha} - Q\} &\cup\\ \{w \in \mathbb{C} : \operatorname{Re}(w) \le Q, \text{ and } |\operatorname{Im} w| \ge -\operatorname{Re}(w) + 2Q\} &\cup\\ \{w \in \mathbb{C} : \operatorname{Re}(w) \ge \frac{1}{\alpha} - Q, \text{ and } |\operatorname{Im} w| \ge \operatorname{Re}(w) - \frac{1}{\alpha} + 2Q\}&.\end{aligned}$$

Lemma 3.3. *For every map $h \in e^{2\pi\alpha \mathbf{i}}\mathcal{IS}$ with α less than ε_0 (obtained in Lemma 3.2), there is a univalent map $L_h\colon \mathrm{Dom}\,(L_h) \to \mathbb{C}$ with the following properties:*

(1) $\Sigma_{C_2} \cup \{\mathrm{cp}_{F_h}\} \subset \mathrm{Dom}\,(L_h)$ *and*

$$\{w \in \mathbb{C} : 0 \le \operatorname{Re}(w) \le \lfloor 1/\alpha \rfloor - k\} \subseteq L_h(\mathrm{Dom}\,(L_h))$$

(same k as in Theorem 1.3).

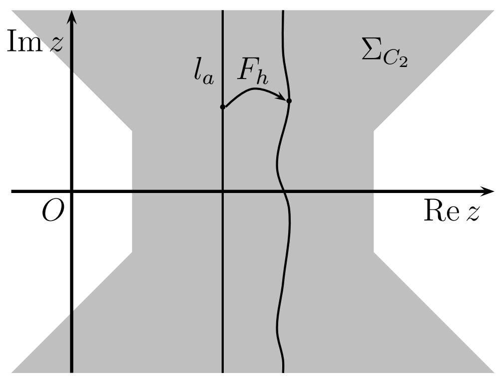

FIGURE 4. The gray region shows the domain Σ_{C_2}.

(2) *L_h satisfies*

(Abel functional equation) $$L_h(F_h(w)) = L_h(w) + 1$$

whenever both sides are defined. Moreover, L_h is unique once normalized by mapping the critical value of F_h to 1.

Proof. Let l_a denote the vertical line $\{a + \mathbf{i}t : -\infty < t < +\infty\}$, for a in $[C_2, \frac{1}{\alpha} - C_2 - \frac{5}{4}]$. If $\alpha \leq \varepsilon_0$, by Lemma 3.2–(1), image of l_a under F_h does not intersect itself. By the same lemma, the two curves l_a and $F_h(l_a)$ cut the complex plane into three connected components. Denote closure of the one with bounded real part by $\mathcal{K}_h$.

Consider the homeomorphism

$$g\colon \{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq 1\} \to \mathcal{K}_h$$

defined as

$$g(s + \mathbf{i}t) := (1 - s)(a + \mathbf{i}t) + sF_h(a + \mathbf{i}t).$$

The partial derivatives of g exist everywhere and can be calculated as

$$\begin{aligned}
\frac{\partial g}{\partial w}(s + \mathbf{i}t) &= \frac{1}{2}[\frac{\partial g}{\partial s} - \mathbf{i}\frac{\partial g}{\partial t}](s + \mathbf{i}t)\\
&= \frac{1}{2}[F_h(a + \mathbf{i}t) - (a + \mathbf{i}t) + 1 + s(F_h'(a + \mathbf{i}t) - 1)],\\
\frac{\partial g}{\partial \bar{w}}(s + \mathbf{i}t) &= \frac{1}{2}[\frac{\partial g}{\partial s} + \mathbf{i}\frac{\partial g}{\partial t}](s + \mathbf{i}t)\\
&= \frac{1}{2}[F_h(a + \mathbf{i}t) - (a + \mathbf{i}t) - 1 + s(1 - F_h'(a + \mathbf{i}t))].
\end{aligned} \tag{7}$$

By the inequalities in part (1) of Lemma 3.2, dilatation of the map g, $|g_{\bar{w}}/g_w|$, is bounded by 1/3. Thus, it is a quasi-conformal map onto $\mathcal{K}_h$. Moreover,

$$\forall w \in l_a,\ g^{-1}(F_h(w)) = g^{-1}(w) + 1.$$

The Beltrami differential

$$\nu(w) := \frac{\partial g/\partial \bar{w}}{\partial g/\partial w}(w)\frac{d\bar{w}}{dw}$$

is the pull back of the standard complex structure on $\mathbb{C}$ by g. Using $\nu(T_1(w)) = \nu(w)$, we can extend $\nu(w)$ over the whole complex plane $\mathbb{C}$. By measurable Riemann mapping theorem ([Ahl06], Ch V, Theorem 3), there exists a unique quasi-conformal mapping $g_1 : \mathbb{C} \to \mathbb{C}$ which solves the Beltrami differential equation $\frac{\partial}{\partial \bar{w}} g_1 = \nu \cdot \frac{\partial}{\partial w} g_1$ and leaves the points 0 and 1 fixed.

As $g_1 \circ T_1 \circ g_1^{-1}$ is quasi-conformal and $\partial(g_1 \circ T_1 \circ g_1^{-1})/\partial \bar{w} = 0$, by explicit calculation, Weyl's Lemma ([Ahl06], Ch II, Corollary 2) implies that this map is a conformal map of the complex plane. As it is conjugate to T_1, it can not have any fixed point. Therefore, it is a translation of the plane. Finally, $g_1(0) = 1$ implies that $g_1 \circ T_1 \circ g_1^{-1} = T_1$, or in other words, $g_1(w + 1) = g_1(w) + 1$.

For the same reason, the map $L_h := g_1 \circ g^{-1} : \mathcal{K}_h \to \mathbb{C}$ is conformal and by previous arguments satisfies $L_h(F_h(w)) = L_h(w) + 1$ on l_a. This relation can be used to extend L_h on a larger domain. By part (1) of Lemma 3.2, for every $w \in \Sigma_{C_2}$ there is an integer j_w for which $F_h^{j_w}(w) \in \mathcal{K}_h$. Thus domain of L_h contains at least Σ_{C_2} and by definition satisfies the Abel functional equation on its domain of definition.

Note that $L_h(a) = 0$. Given any simply connected domain in $\mathbb{C} \setminus \{0, \sigma_h\}$, there is a continuous inverse branch of τ_h defined on this domain. Further, if image of such a domain under this branch, τ_h^{-1}, is contained in domain of L_h, composition of this branch and L_h is a Fatou coordinate for h. By uniqueness in Theorem 1.3, there exists a constant t_h such that $L_h \circ \tau_\alpha^{-1} + t_h = \Phi_h$ is the unique Fatou coordinate which sends cp_h to zero. This would imply that L_h extends over a larger domain containing cp_{F_h} on its boundary. Moreover, its image contains the set

$$\{w \in \mathbb{C} : 0 \le \mathrm{Re}(w) \le \lfloor 1/\alpha \rfloor - k\}$$

for the constant k obtained in that theorem. □

To control Fatou coordinate of a given map h, which is of the from $L_h \circ \tau_h^{-1}$, we need to control L_h. First we give a rough estimate on derivative of L_h.

Lemma 3.4. *There exists a positive constant C_5 such that for every $h \in e^{2\pi\alpha \mathbf{i}}\mathcal{IS}$, and every ζ with $1 \le \mathrm{Re}\,\zeta \le 1/\alpha - k$, we have $1/C_5 \le |(L_h^{-1})'(\zeta)| \le C_5$.*

Proof. Let $G : (0, 1/\alpha - k) \times (-\infty, \infty) \to \mathbb{C}$, denote the map L_h^{-1} through this proof. We will consider two separate cases. First assume $\xi := G(\zeta) \in \Theta(C_2)$ and $\mathrm{Im}\,\zeta \in (1.5, 1/\alpha - k - 1.5)$. So, G is defined and univalent on $B(\zeta, 1.5)$ and $F_h(\xi) \in B(\xi + 1, 1/4)$. Now, by 1/4 Theorem, $|G'(\zeta)|/4 \le (1 + 1/4)$ which implies $G'(\zeta) \le 5$. For the other direction, by Koebe distortion theorem, we have

$$\forall w \in B(\zeta, 1),\ |G'(w)/G'(\zeta)| \le 45.$$

By comparing distances $d(\zeta, \zeta+1)$ and $d(\xi, F_h(\xi))$, we obtain

$$45|G'(\zeta)| \geq 1 \cdot \sup_{w \in B(\zeta,1)} |G'(w)| \geq 1 - 1/4 = 3/4,$$

which implies, $|G'(\zeta)| \geq 1/36$. This proves the lemma in this case.

Now if $\xi \in \Theta(C_2)$ and $\operatorname{Im} L_h(\xi) \in (1, 1/\alpha - k)$. By Abel functional equation in Lemma 3.3, at least one of ξ, $F_h(\xi)$, $F_h^2(\xi)$, $F_h^{-1}(\xi)$, $F_h^{-2}(\xi)$ satisfies above condition. Differentiating Abel functional equation and using Lemma 3.2, part (1), we see

$$3/4 \leq |L_h'(F_h(\xi))|/|L_h'(\xi)| = |F_h'(\xi)| \leq 5/4,$$

which takes care of this case.

Finally, if $\xi \notin \Theta(C_2)$ then ξ belong to a compact subset of $\mathbb{C}$. As the normalized Fatou coordinate L_h is univalent and depends continuously on h in the compact open topology, this case follows from compactness of the class $\mathcal{IS}[\alpha_*]$. □

Finally, the following is a fine control of L_h. Let $C_6 > 1$ be a positive constant that satisfies $C_4(1 + \log \frac{5}{4\alpha}) + 2C_5 \leq C_6/\alpha$.

Lemma 3.5. *There exists a positive constant C_7 such that for every map L_h with $\alpha(h) < \varepsilon_0$, every $r \in (0, 1/2)$, and every $w_1, w_2 \in \operatorname{Dom} L_h$ with*

- $\operatorname{Re} w_1 = \operatorname{Re} w_2$, *and* $\operatorname{Im} w_1, \operatorname{Im} w_2 > -C_6/\alpha$,
- *for all* $t \in (0,1)$, $tw_1 + (1-t)w_2 \in \Theta(\frac{r}{\alpha} + 1)$,

we have,

(1) $|\operatorname{Re}(L_h(w_1) - L_h(w_2))| \leq C_7/r$
(2) $|\operatorname{Im}(L_h(w_1) - L_h(w_2)) - \operatorname{Im}(w_1 - w_2)| \leq C_7/r$

Proof. Given w_1, w_2 satisfying the conditions in the lemma, choose a vertical line l with $w_1, w_2 \in \mathcal{K}_h$ in the construction of the map g in Lemma 3.3. Let $F_h^{i(h)}(\mathrm{cp}_{F_h})$ be the first visit of cp_{F_h} to Σ_{C_2}, and let $w^* := F_h^{i(h)+j}(\mathrm{cp}_{F_h})$ be the first visit of this point to $\mathcal{K}_h$. By part (1) of Lemma 3.2, $j \leq \frac{1/2\alpha}{3/4} \leq \frac{2}{3\alpha}$. Hence, (4) of the same lemma implies that

$$\operatorname{Im} w^* \in [-C_4(1 + \log \frac{2}{3\alpha}), C_4(1 + \log \frac{2}{3\alpha})]. \tag{8}$$

Let t_h be a complex constant with imaginary part in this set and

$$(L_h + t_h)(w^*) = i(h) + j.$$

Then by Abel functional equation we conclude that cp_{F_h} is mapped to 0 under $L_h + t_h$. We will denote this map by the same notation L_h, thus $L_h(w^*) = i(h) + j$.

We have the following simple inequalities for the quasi-conformal map g^{-1} constructed using the choice of vertical line l:

$$|\operatorname{Im}(g^{-1}(w_1) - g^{-1}(w_2)) - \operatorname{Im}(w_1 - w_2)| \leq 1/2,$$
$$|\operatorname{Re}(g^{-1}(w_1)) - \operatorname{Re}(g^{-1}(w_2))| \leq 1/2.$$

To prove similar results for L_h, we will compare it to g^{-1} using Green's integral formula. Choose t_1 and t_2 so that w_1 and w_2 are contained in the curves $s \mapsto g(s + \mathbf{i}t_1)$, and

$s \mapsto g(s + \mathbf{i}t_2)$, for $0 \le s \le 1$, respectively. Using notations $\zeta = s + \mathbf{i}t$, $d\zeta = ds + \mathbf{i}dt$ and $d\bar{\zeta} = ds - \mathbf{i}dt$, by Green's Theorem applied to the map $g_1(\zeta) = L_h \circ g$ on the rectangle

$$\mathcal{D} := \{\zeta \in \mathbb{C} : 0 \le \mathrm{Re}(\zeta) \le 1,\ t_1 \le \mathrm{Im}\,\zeta \le t_2\}$$

we have

$$\text{(Green's formula)} \qquad \int_{\partial\mathcal{D}} g_1(\zeta)\, d\zeta = \iint_{\mathcal{D}} -\frac{\partial g_1(\zeta)}{\partial\bar{\zeta}}\, d\zeta \wedge d\bar{\zeta}.$$

If we let $w = g(\zeta)$, the complex chain rule for $g_1(\zeta)$, using the Cauchy-Riemann equation $\frac{\partial L_h}{\partial \bar{w}} = 0$, can be written as

$$\frac{\partial g_1}{\partial\bar{\zeta}} = \frac{\partial(L_h \circ g)}{\partial\bar{\zeta}} = (\frac{\partial L_h}{\partial w} \circ g)\frac{\partial g}{\partial\bar{\zeta}}.$$

Therefore,

$$\begin{aligned}\Big|\iint_{\mathcal{D}} \frac{\partial g_1(\zeta)}{\partial\bar{\zeta}}\, d\zeta \wedge d\bar{\zeta}\Big| &\le \int_{t_1}^{t_2}\int_0^1 2\Big|\frac{\partial g_1(\zeta)}{\partial\bar{\zeta}}\Big|\, dsdt \\ &\le \int_{t_1}^{t_2}\int_0^1 \frac{4}{3}\cdot \sup|L_h'| \cdot C_3\frac{\alpha}{r}e^{-2\pi\alpha\,\mathrm{Im}\, g(s+\mathbf{i}t)}\, dsdt.\end{aligned}$$

The last inequality follows from (7) and Lemma 3.2–(2). By our assumption on w_1 and w_2, the last integral is less than or equal to

$$\begin{aligned}&\int_{-C_6/\alpha}^{\infty} \frac{4}{3}C_5C_3\frac{\alpha}{r}e^{-2\pi\alpha(t-1/4)}\, dt \\ &\le \frac{4C_5C_3}{3\pi r}e^{-2\pi\alpha(-C_6/\alpha-1/4)} \\ &\le \frac{4C_5C_3e^{\pi(2C_6+1)}}{3\pi}\frac{1}{r},\end{aligned}$$

which is bounded independent of α.

If we parametrize boundary of $\mathcal{D}$ as

$$\begin{aligned}&\vartheta_1(\ell) := \mathbf{i}\ell,\ \ell \in [t_1, t_2] && \vartheta_2(\ell) := \ell + \mathbf{i}t_2,\ \ell \in [0,1] \\ &\vartheta_3(\ell) := 1 + \mathbf{i}(t_1 + t_2 - \ell),\ \ell \in [t_1, t_2] && \vartheta_4(\ell) := 1 - \ell,\ \ell \in [0,1]\end{aligned}$$

the left hand side of the (Green's formula) can be written as

$$\begin{aligned}&\int_{t_1}^{t_2} g_1(\mathbf{i}\ell)\mathbf{i}\, d\ell + \int_0^1 g_1(\ell + \mathbf{i}t_2)\, d\ell + \\ &\qquad \int_{t_1}^{t_2} g_1(1 + \mathbf{i}(t_1 + t_2 - \ell))(-\mathbf{i})\, d\ell + \int_0^1 -g_1(1-\ell)\, d\ell.\end{aligned}$$

Replacing $g_1(\zeta+1)$ by $g_1(\zeta)+1$ and making a change of coordinate in the third integral, we obtain

$$-i(t_2-t_1)+\int_0^1 g_1(\ell+\mathbf{i}t_2)\,d\ell+\int_0^1 -g_1(1-\ell)\,d\ell.$$

Now we show that the above two integrals are in bounded distance of $L_h(w_2)$ and $-L_h(w_1)$, as follows:

$$\begin{aligned}\Big|\int_0^1 g_1(\ell+\mathbf{i}t_2)\,d\ell-L_h(w_2)\Big| &\leq \int_0^1 |g_1(\ell+\mathbf{i}t_2)-L_h(w_2)|\,d\ell\\ &=\int_0^1 |g_1(\ell+\mathbf{i}t_2)-g_1(\ell_1+\mathbf{i}t_2)|\,d\ell,\\ &\leq \int_0^1 \sup_{\zeta\in[0,1]+\mathbf{i}t_2}|g_1'(\zeta)|\,d\ell\leq \frac{5}{4}C_5,\end{aligned}$$

for some $\ell_1\in[0,1]$. Similarly

$$\Big|\int_0^1 -g_1(1-\ell)\,d\ell+L_h(w)\Big|\leq \frac{5}{4}C_5.$$

Now one infers parts (1) and (2) of the lemma by considering real part and imaginary part of Green's formula). □

3.4. **Proof of the main technical lemmas.**

Proof of Lemma 2.6. It is enough to prove the statement for small values of α_n. That is because the sector $f_n^{k_n}(S_n^0)$ is contained in Dom $f_n\subset B(0,4/27e^{4\pi})$. Therefore, it has uniformly bounded diameter. Now, one can choose a large enough M to satisfy the inequality in the lemma once α_n is not too small.

Let L_n denote the linearizing map L_{f_n} corresponding to f_n that can be obtained in Lemma 3.3. Consider the half-line

$$\gamma(t):=F_n^{\lfloor 1/2\alpha_n\rfloor}(\mathrm{cp}_{F_n})+\mathbf{i}t:[-1/\alpha_n-4C_7,\infty)\to\mathbb{C}.$$

By part (4) of Lemma 3.2,

$$\operatorname{Re}F_n^{\lfloor 1/2\alpha_n\rfloor}(\mathrm{cp}_{F_n})\in[\lfloor 1/2\alpha_n\rfloor-C_4(1+\log\frac{1}{2\alpha_n}),\lfloor 1/2\alpha_n\rfloor+C_4(1+\log\frac{1}{2\alpha_n})].$$

Thus, for sufficiently small α_n, one can use Lemma 3.5, with $r=1/4$, $w_1=F_n^{\lfloor 1/2\alpha_n\rfloor}(\mathrm{cp}_{F_n})$ and w_2 any point on γ, to conclude that

$$\operatorname{diam}\{\operatorname{Re}L_n(\gamma(t))-\frac{1}{2\alpha_n}:t\in\mathrm{Dom}\ \gamma\}\leq 4C_7,$$
$$\operatorname{Im}L_n(\gamma(-1/\alpha_n-4C_7))\leq -1/\alpha_n.$$

Hence, the set

$$\bigcup_{t\in\mathrm{Dom}\ \gamma} B(L_n(\gamma(t)),4C_7+2)$$

contains the half-strip

$$A := \{\zeta \in \mathbb{C} : \lfloor \frac{1}{2\alpha_n} \rfloor - 1/2 \leq \operatorname{Re} \zeta \leq \lfloor \frac{1}{2\alpha_n} \rfloor + 1/2, \operatorname{Im} \zeta \geq -\frac{1}{\alpha_n}\}. \tag{9}$$

Now, by Lemma 3.4, image of this strip under L_n^{-1} must be contained in the set

$$\bigcup_{t \in \operatorname{Dom} \gamma} B(\gamma(t), C_5(4C_7 + 2)),$$

which is, by Lemma 3.2 part (4), a subset of the half-strip

$$B := \Big\{\zeta \in \mathbb{C} : |\operatorname{Re} \zeta - \lfloor \frac{1}{2\alpha_n} \rfloor| \leq C_4(1 + \log \frac{1}{2\alpha_n}) + C_5(4C_7 + 2),$$
$$\operatorname{Im} \zeta \geq \frac{-1}{\alpha_n} - 4C_7 - C_4(1 + \log \frac{1}{2\alpha_n}) - C_5(4C_7 + 2)\Big\}.$$

By definition of S_n^0,

$$f_n^{k_n}(S_n^0) = \{z \in \mathcal{P}_n : 1/2 \leq \operatorname{Re} \Phi_n(z) \leq 3/2, \operatorname{Im} \Phi_n(z) \geq -2\}.$$

Equivariance relation, (Theorem 1.3–b)), implies that

$$f_n^{k_n + \lfloor 1/2\alpha_n \rfloor - 1}(S_n^0)$$
$$= \{z \in \mathcal{P}_n : \lfloor \frac{1}{2\alpha_n} \rfloor - 1/2 \leq \operatorname{Re} \Phi_n(z) \leq \lfloor \frac{1}{2\alpha_n} \rfloor + 1/2, \operatorname{Im} \Phi_n(z) \geq -2\}.$$

Since $\Phi_n^{-1} = \tau_n \circ L_n^{-1}$, to conclude the lemma, It is enough to bound diam $\tau_n(B)$. For small α_n, Lemma 3.1 with $r = 1/4$ applies and we obtain diam $\tau_n(B) \leq M\alpha_n$, where $M = 4C_1 e^{\pi(2+4C_7+C_5(4C_7+2)+3C_4)}$. We have further shown that:

$$\forall \zeta \in A, \;\; |\tau_n(\zeta)| \leq M\alpha_n \tag{10}$$

Which will be used later. □

Proof of Lemma 2.7. If α_n is large and $|w|$ is also bounded below then one can make choose the constant M large enough. So we only consider other cases.

First assume that α_n is small so that the following argument works. Recall that η_{n+1} is an arbitrarily chosen inverse branch of $\mathbb{E}\mathrm{xp}$ on $\mathcal{P}_{n+1}$. So we may assume that $\operatorname{Re}(\eta_{n+1}(\mathcal{P}_{n+1}) \subset [0, \hat{k}]$ by Theorem 1.3. If we let $\zeta = \eta_{n+1}(w)$, then $\operatorname{Im} \zeta = \frac{-1}{2\pi} \log \frac{27|w|}{4}$. Now, let $\kappa(n) := \lfloor \frac{1}{2\alpha_n} \rfloor$.

It follows from Lemma 3.2 and 3.5 with $r = 1/4$ (for small enough α_n) that $L_n^{-1}(\zeta + \kappa(n))$ satisfies the following:

$$\frac{1}{4\alpha_n} \leq \operatorname{Re} L_n^{-1}(\zeta + \kappa(n)) \leq \frac{3}{4\alpha_n},$$
$$\operatorname{Im} L_n^{-1}(\zeta + \kappa(n)) \geq \frac{-1}{2\pi} \log \frac{27|w|}{4} - 4C_7 - C_4(1 + \log \frac{1}{2\alpha_n}).$$

Now one uses Lemma 3.1 part (2), with $r = 1/4$, to obtain

$$\begin{aligned}|f_n^{\kappa(n)}(\psi_{n+1}(w))| &= |\tau_n(L_n^{-1}(\zeta+\kappa(n)))|\\ &\le 4C_1\alpha_n e^{-2\pi\alpha_n \operatorname{Im} L_n^{-1}(\zeta)}\\ &\le 27C_1 e^{(4C_7+2)\pi}\cdot \alpha_n |w|^{\alpha_n}.\end{aligned}$$

which proves the lemma in this case.

The lemma for larger α_n and sufficiently small $|w|$ follows from compactness of the class $\mathcal{IS}[\alpha_*]$. Indeed, f_n belongs to the class $\mathcal{IS}[\alpha_*]$ with $\alpha_n \in [\varepsilon, \alpha^*]$ for some ε and one can see that the associated map F_n converges geometrically to $z \to z+1$ as $\operatorname{Im} z \to \infty$. This implies that the linearizing map L_n is bounded away from a translation at points with large imaginary part. Now one uses continuous dependence of linearizing map L_h^{-1} on the map F_h on a compact set $[\varepsilon, \alpha^*]\times[-2, \text{large number}]$ to conclude that the translation constant must have a bounded absolute value. Therefore, if $\zeta = \eta_{n+1}(w)$, then $|\operatorname{Im} L_n^{-1}(\zeta) - \operatorname{Im}\zeta| \le M'$ for some constant M'. Like above argument this implies that for any choice of $\kappa(n) \in [0, \frac{1}{\alpha_n} - k - 1]$ we have

$$|f_n^{\kappa(n)} \circ \psi_{n+1}(w)| \le M'' \cdot \alpha_n \cdot |w|^{\alpha_n},$$

for some constant M''. □

4. Measure and topology of the attractor

In this section we consider Lebesgue measure (area) and topology of the post-critical set of quadratic polynomials with a non-Brjuno multiplier of high return times.

We will show that intersection of the sets Ω_0^n, which contains the post-critical set by Proposition 2.4, has area zero, by showing that it does not contain any Lebesgue density point. Strategy of our proof is to show that given any point z in this intersection, one can find balls of arbitrarily small size but comparable to their distance to z in the complement of the intersection. The balls will be introduced in domains of the renormalized maps f_n and then transferred through our changes of coordinates to the dynamic plane of P_α. We will use the Koebe distortion theorem to derive required properties about shape, size, and the distance to z of the image balls.

4.1. **Balls in the complement.** The following lemma guarantees the complementary balls within the domain of each f_n.

Lemma 4.1. *There are positive constants δ_1 and r^* such that for every $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \le \frac{1}{2\pi}\log\frac{1}{\alpha_{n+1}}$, and $\mathbb{E}\mathrm{xp}(\zeta) \in \Omega_{n+1}^0$ for some integer $n \ge 1$, there exists a line segment $\gamma_n : [0,1] \to \mathbb{C}$ with $\gamma_n(0) = \zeta$ satisfying the following properties:*

1. $\mathbb{E}\mathrm{xp}\big(B(\gamma_n(1), r^*)\big) \cap \Omega_{n+1}^0 = \varnothing$, $f_{n+1}\big(\mathbb{E}\mathrm{xp}(B(\gamma_n(1), r^*))\big) \cap \Omega_{n+1}^0 = \varnothing$,
2. $\mathbb{E}\mathrm{xp}\big(B_{\delta_1}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1])\big) \subseteq \mathrm{Dom}\,(f_{n+1}) \setminus \{0\}$,
3. $\mathrm{diam}\ \mathrm{Re}\big(B_{\delta_1}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1])\big) \le 1 - \delta_1$,
4. $\mathrm{mod}\ B_{\delta_1}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1]) \setminus (B(\gamma_n(1), r^*) \cup \gamma_n) \ge \delta_1$.

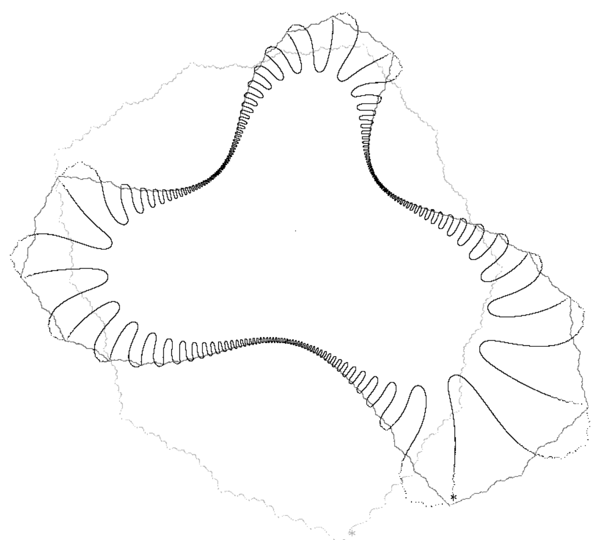

FIGURE 5. The three curves in different colors approximate part of the orbit of the critical point for different values of α. The light grey one is for $\alpha = [3, 1, 1, 1, \ldots]$, the grey one for $[3, 50, 1, 1, 1, \ldots]$, and the dark grey one for $[3, 50, 10^5, 1, 1, 1, \ldots]$.

Proof. First assume $\alpha_{n+1} \leq \min\{\frac{1}{k''+k}, \frac{1}{8(k+1)}, \varepsilon_0\}$, where ε_0 is the constant obtained in Lemma 3.2. Consider the line segment

$$\vartheta(t) := t - (2 + t/2)\mathbf{i} : [2, \frac{1}{2\alpha_{n+1}}] \to \mathbb{C}$$

between the two points $2 - 3\mathbf{i}$, and $\frac{1}{2\alpha_{n+1}} - (2 + \frac{1}{4\alpha_{n+1}})\mathbf{i}$. For every $t \in [2, \frac{1}{2\alpha_{n+1}}]$, under our assumption $\alpha_{n+1} \leq \frac{1}{4(k+1)}$, we have

$$\begin{aligned} &B(\vartheta(t), t/2) \subset \{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq \frac{1}{\alpha_{n+1}} - k, \mathrm{Im}\, w \leq -2\}, \\ &B(\vartheta(t), t/2) + 1 \subset \{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq \frac{1}{\alpha_{n+1}} - k, \mathrm{Im}\, w \leq -2\}. \end{aligned} \tag{11}$$

Similarly (when $\alpha_{n+1} \leq 1/8k$) one can see that

$$B(\vartheta(t), 3t/4) \subset \{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq \frac{1}{\alpha_{n+1}} - k\} = \Phi_{n+1}(\mathcal{P}_{n+1})$$

which gives the following lower bound for conformal modulus:

$$\mathrm{mod}\ (\Phi_{n+1}(\mathcal{P}_{n+1}) \setminus B(\vartheta(t), t/2)) \geq \frac{1}{2\pi} \log \frac{3}{2}. \tag{12}$$

The idea of the proof is to show that lifts of $\Phi_{n+1}^{-1}(B(\vartheta(t), t/2))$ via $\mathbb{E}\mathrm{xp}$ provide balls satisfying the required properties in the lemma. First we will consider lifts of the curve

$\Phi_{n+1}^{-1} \circ \vartheta$, via $\mathbb{E}\mathrm{xp}$ and show that they start from a bounded height and reach the needed height $\frac{1}{2\pi} \log \frac{1}{\alpha_{n+1}}$. Then we consider lifts of $\Phi_{n+1}^{-1}(B(\vartheta(t), t/2))$ and show that they contain balls of a definite size.

Recall that $\Phi_{n+1}^{-1} = \tau_{n+1} \circ L_{n+1}^{-1}$. By Lemma 3.4 we have

$$\begin{aligned} |L_{n+1}^{-1}(\vartheta(2)) - cv_{F_{n+1}}| &\leq \sup |L_{n+1}^{-1}| \cdot |(2,-3) - (1,0)| \\ &\leq C_5\sqrt{10}. \end{aligned}$$

Since τ_{n+1} maps the critical value $cv_{F_{n+1}}$ to $-4/27$, one can see that every point in $\mathbb{E}\mathrm{xp}^{-1}(\Phi_{n+1}^{-1}(\vartheta(2)))$ has imaginary part uniformly bounded above by some constant δ.

For the other end point, $\vartheta(\frac{1}{2\alpha_{n+1}})$ belongs to the half-strip A defined in (9). Thus by (10), we have

$$|\Phi_{n+1}^{-1}(\vartheta(\frac{1}{2\alpha_{n+1}}))| \leq M\alpha_{n+1}.$$

This implies that every point in $\mathbb{E}\mathrm{xp}^{-1}(\Phi_{n+1}^{-1}(\vartheta(\frac{1}{2\alpha_{n+1}})))$ has imaginary part bigger than $\frac{1}{2\pi} \log \frac{1}{\alpha_{n+1}} - \frac{1}{2\pi} \log \frac{27M}{4}$.

To transfer the balls, consider the map

$$\eta_{n+1} \circ \tau_{n+1} \circ L_{n+1}^{-1} : \Phi_{n+1}(\mathcal{P}_{n+1}) \to \mathbb{C} \tag{13}$$

where η_{n+1} is an arbitrary inverse branch of $\mathbb{E}\mathrm{xp}$ defined on $\mathbb{C}$ minus a ray landing at 0.

We claim that there exists a constant M' such that derivative of the above map at every point $t - 2\mathbf{i} \in \mathbb{C}$, $t \in \operatorname{Dom} \vartheta$, is at least M'/t. By compactness of the class $\mathcal{IS}[\alpha_*]$ and continuous dependence of linearizing map in the compact-open topology it is enough to prove the claim for values of t bigger than some constant (indeed when $C_4(1+\log t) \leq t/2$)). Also by Koebe distortion theorem, it is enough to prove this for integer values of t bigger than that constant. For such t's, $L_{n+1}^{-1}(t) = F_{n+1}^t(cv_{F_{n+1}})$. So by Lemma 3.2 part (4), we have

$$|\operatorname{Im} L_{n+1}^{-1}(t)| \leq C_4(1 + \log t), \text{ and } \quad |\operatorname{Re} L_{n+1}^{-1}(t) - t| \leq C_4(1 + \log t).$$

Hence, by Lemma 3.4,

$$\begin{aligned} &|\operatorname{Im} L_{n+1}^{-1}(t - 2\mathbf{i})| \leq C_4(1 + \log t) + 2C_5, \\ &|\operatorname{Re} L_{n+1}^{-1}(t - 2\mathbf{i}) - t| \leq C_4(1 + \log t) + 2C_5. \end{aligned} \tag{14}$$

Define the set O_t as follows:

$$O_t := \{\xi \in \mathbb{C} : |\operatorname{Im} \zeta| \leq C_4(1 + \log t) + 2C_5, \text{ and } |\operatorname{Re} \zeta - t| \leq C_4(1 + \log t) + 2C_5\}$$

The point t belongs to O_t, and by an explicit calculation one can see than

$$(\eta_{n+1} \circ \tau_{n+1})'(t) \geq 1/2t.$$

As mod $(\mathbb{C}^* \setminus O_t)$ is bounded below independent of t (indeed, it is increasing in terms of t), Koebe distortion theorem implies that there exists a constant M'' such that for every $\xi \in O_t$, we have

$$(\eta_{n+1} \circ \tau_{n+1})'(\xi) \geq M''/t.$$

Since $L_{n+1}^{-1}(t-2\mathbf{i}) \in O_t$, by (14), combining with Lemma 3.4 we have

$$(\eta_{n+1} \circ \tau_{n+1} \circ L_{n+1}^{-1})'(t-2\mathbf{i}) \geq \frac{M''}{C_5}\frac{1}{t}.$$

Again Koebe distortion theorem, using (12), implies that

$$\forall \xi \in B(\vartheta(t), t/2) \text{ we have,} \quad (\eta_{n+1} \circ \tau_{n+1} \circ L_{n+1}^{-1})'(\xi) \geq M'/t$$

for some constant M' independent of t and α_{n+1}. Therefore, image of the ball $B(\vartheta(t), t/2)$ under the map (13) contains a ball of constant radius r^* around $\mathbb{E}\mathrm{xp}$ pre-images of $\Phi_{n+1}^{-1}(\vartheta(t))$.

Domain of f_{n+1} contains the ball of radius .22, therefore, every point in $\mathbb{C}$ with positive imaginary part is mapped into Dom f_{n+1} under $\mathbb{E}\mathrm{xp}$. To associate a curve γ_n to the given point ζ, we consider the following two separate cases:

If $\operatorname{Im}\zeta \geq 1$, by previous argument, there exists a point ζ' in a lift of $\Phi_{n+1}^{-1}(\vartheta)$ (under $\mathbb{E}\mathrm{xp}$) which satisfies $\operatorname{Re}(\zeta-\zeta') \leq 1/2$ and $\operatorname{Im}(\zeta-\zeta') \leq \max\{\delta, \frac{1}{2\pi}\log\frac{27M}{4}\}$. Define $\gamma_n : [0,1] \to \mathbb{C}$ as the straight line segment between ζ and ζ' with $\gamma_n(0) = \zeta$ and $\gamma_n(1) = \zeta'$. Thus, $B(\gamma_n(1), r^*) \cup \gamma_n[0,1]$ projects into Dom f_{n+1}. Moreover, if r^* is chosen less than 1/4, we have

$$\operatorname{diam}\left(\operatorname{Re}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1])\right) \leq 3/4.$$

Hence, $\mathbb{E}\mathrm{xp}$ is univalent on the 1/4 neighborhood of $B(\gamma_n(1), r^*) \cup \gamma_n[0,1]$.

Part (1) of the lemma follows from (11) and that (when $\alpha_{n+1} \leq \frac{1}{k''+k}$)

$$\bigcup_{j=0}^{k_{n+1}-1} f_{n+1}^{j}(S_{n+1}^0) \cap \{w \in \mathcal{P}_{n+1} : \operatorname{Im}\Phi_{n+1}(w) < -2\} = \varnothing.$$

Parts (2) and (3) follows form definition.

Now assume that $\operatorname{Im}\zeta$ is uniformly bounded above, or α_{n+1} is bounded below (which implies $\operatorname{Im}\zeta$ is bounded above). In this case our argument is based on the compactness of the class $\mathcal{IS}[\alpha_*]$. Indeed, there exists a $\delta' > 0$ such that

$$B_{\delta'}(\Omega_{n+1}^0) \subset \text{Dom } f_{n+1}. \tag{15}$$

Since $\xi = \mathbb{E}\mathrm{xp}(\zeta)$ is away from 0 and has uniformly bounded diameter, there exists a (uniformly bounded) real number $s > 1$, with

$$B(s\xi, \delta'/2) \cap \Omega_{n+1}^0 = \varnothing, \text{ and } f_{n+1}(B(s\xi, \delta'/2)) \cap \Omega_{n+1}^0 = \varnothing.$$

Now, define $\gamma'(t) := t\xi + (1-t)s\xi : [0,1] \to \text{Dom } f_{n+1}$. The curve γ_n is defined as the lift of γ' starting at ζ. Thus, $\mathbb{E}\mathrm{xp}^{-1}(B(s\xi, \delta'/2))$ contains a ball of radius r^* satisfying the lemma in this case as well. □

Lemma 4.2. *There exists a real constant $\delta_2 \leq \delta_1$ such that for every $\xi \in \mathbb{C}$ with $\mathbb{E}\mathrm{xp}(\xi) \in \Omega_{n+1}^0$, we have*

- $\mathbb{E}\mathrm{xp}(B(\xi, \delta_2)) \subset \mathrm{Dom}\,(f_{n+1})$,
- $\forall n \in \mathbb{Z},\ \mathbb{E}\mathrm{xp}(B(n, \delta_2)) \subset \operatorname{int} f_n^{k_n}(S_n^0) \subset \Omega_n^0$.

Proof. It follows from continuous dependence of the Fatou coordinate in the compact-open topology, that there exists a real constant $\delta > 0$ such that for every $n \geq 0$,

$$B(-4/27, \delta) \subset f_n^{k_n}(S_n^0) = \{\xi \in \mathbb{C} : \operatorname{Im} \xi > -2, 1/2 \leq \operatorname{Re} \xi \leq 3/2\}.$$

The first inclusion in the lemma follows from (15) and the second one follows from above observation. $\square$

For every integer $n \geq 1$ and every integer j, with $0 \leq j < \frac{1}{\alpha_n} - k$, we define curves $I_{n,j}$ as follows

$$I_{n,j} := \Phi_n^{-1}\{\xi \in \mathbb{C} : \operatorname{Re} \xi = j,\ \operatorname{Im} \xi > -2\}.$$

Each $I_{n,j}$ is a smooth curve contained in Ω_n^0, and connects boundary of Ω_n^0 to 0. Also one can see that for every such n and j, every closed loop (image of a continuous curve with the same initial and terminal point) contained in $\Omega_n^0 \setminus I_{n,j}$ is contractible in $\mathbb{C}^*$. This implies that there is a continuous inverse branch of $\mathbb{E}\mathrm{xp}$ defined on every $\Omega_n^0 \setminus I_{n,j}$.

By compactness of the class $\mathcal{IS}[\alpha_*]$ there exists a positive integer k' such that

$$\forall j \text{ with } 0 \leq j < \tfrac{1}{\alpha_n} - k, \quad \sup_{z \in \Omega_n^0 \setminus I_{n,j}} \arg(z) \leq 2\pi k', \tag{16}$$

for every continuous branch of argument defined on $\Omega_n^0 \setminus I_{n,j}$. We assume the following technical condition on α_n's

$$\alpha_n \leq \frac{1}{2k' + k} \tag{17}$$

during this section.

4.2. **Going down the renormalization tower.** Fix an arbitrary point z_0 in $\bigcap_{n=0}^{\infty} \Omega_0^n$ different from 0. We associate a sequence of quadruples

$$\{(z_i, w_i, \zeta_i, \sigma(i))\}_{i=0}^{\infty} \tag{18}$$

to z_0, where z_i and w_i are points in $\operatorname{Dom}(f_i)$, ζ_i is a point in $\Phi_i(\mathcal{P}_i)$ and $\sigma(i)$ is a non-negative integer. This sequence will serve as a guide to transfer the balls in the previous lemma to the dynamic plane of f_0.

The sequence of quadruples (18) is defined inductively as follows: Since $z_0 \in \cup_{j=0}^{1/\alpha_0 - k + k_n - 1} f_0^j(S_0^0)$, we have one of the following two possibilities:

Case I: $z_0 \in \mathcal{P}_0$, and one of the following two occurs:

– $\operatorname{Re} \Phi_0(z_0) \in [k' + 1/2, 1/\alpha_0 - k]$,

– $\Phi_0(z_0) \in B(j, \delta_2)$ for some $j = 1, 2, \ldots, k'$

Case II: $z_0 \in \mathcal{P}_0$, $\operatorname{Re} \Phi_0(z_0) \in [0, k' + 1/2)$, and $\Phi_0(z_0) \notin B(j, \delta_2)$, for $j = 1, 2, \ldots, k'$.
Or,
$z_0 \notin \mathcal{P}_0$.

If case I occurs, define $w_0 := z_0$, $\sigma(0) := 0$, and $\zeta_0 := \Phi_0(w_0)$.

If case II occurs, let $w_0 \in S_0^0$, and positive integer $\sigma(0) \leq k_0 + k'$ be such that $f_0^{\sigma(0)}(w_0) = z_0$. The point w_0 satisfying this property is not necessarily unique, however, one can take any of them. The positive integer $\sigma(0)$ is uniquely determined. Indeed when $\sigma(0) \leq k_0 - 1$ or $|z_0|$ is small enough, such w_0 is unique , otherwise, there are at most two choices for w_0. The point ζ_0 is defined as $\Phi_0(w_0)$. This defines the first quadruple $(z_0, w_0, \zeta_0, \sigma(0))$.

Now, let $z_1 := \mathbb{E}\mathrm{xp}(\zeta_0)$. Since z_0 belongs to Ω_0^1, One can see that z_1 belongs to $\cup_{j=0}^{1/\alpha_1 - k + k_1 - 1} f_1^j(S_1^0)$. Thus, we can repeat the above process (replacing 0 by 1 in the above cases) to define the quadruple $(z_1, w_1, \zeta_1, \sigma(1))$ and so on. In general, for every $l \geq 0$,

$$
\begin{aligned}
& z_l = \mathbb{E}\mathrm{xp}(\zeta_{l-1}),\ z_l \in \cup_{j=0}^{1/\alpha_l - k + k_l - 1} f_l^j(S_l^0), \\
& f_l^{\sigma(l)}(w_l) = z_l,\ \Phi_l(w_l) := \zeta_l, \\
& 0 \leq \sigma(l) \leq k_l + k' \leq k'' + k'
\end{aligned}
\tag{19}
$$

where k'' is a uniform bound on the integers k_l.

By definition of this sequence, for every $n \geq 0$ we have

$$
\begin{aligned}
& k' + 1/2 \leq \operatorname{Re} \zeta_n \leq \frac{1}{\alpha_n} - k, \text{ or,} \\
& \zeta_n \in B(j, \delta_2), \text{ for some } j \in \{1, 2, \dots, k'\}.
\end{aligned}
\tag{20}
$$

The following lemma guarantees that some of ζ_j in the above sequence reach the balls provided in the previous lemma.

Lemma 4.3. *Assume that $z_0 \in \cap_{n=0}^{\infty} \Omega_0^n \setminus \{0\}$, and α is a non-Brjuno number in Irr_N. If $\{\zeta_j\}_{j=0}^{\infty}$ is the above sequence associated to z_0, then there are arbitrarily large positive integers m with*

$$\operatorname{Im} \zeta_m \leq \frac{1}{2\pi} \log \frac{1}{\alpha_{m+1}}.$$

To see this, we need the following lemma. Let D_1 be a constant such that

$$\frac{D_1}{\alpha_{n+1}} \geq \frac{1}{4\alpha_{n+1}} + C_2 + 4C_7 + C_4(1 + \log \frac{1}{\alpha_{n+1}})$$

for every $\alpha_{n+1} \in [1/2, \infty)$, where the constants C_2,C_4, and C_7 were introduced in Lemmas 3.2 and 3.5.

Lemma 4.4. *There exists a positive constant D_2 such that for every $n > 0$, we have*

$$
\begin{aligned}
& \textit{if } \operatorname{Im} \zeta_{n+1} \geq \frac{D_1}{\alpha_{n+1}}, \\
& \qquad \textit{then } \operatorname{Im} \zeta_{n+1} \leq \frac{1}{\alpha_{n+1}} \operatorname{Im} \zeta_n - \frac{1}{2\pi\alpha_{n+1}} \log \frac{1}{\alpha_{n+1}} + \frac{D_2}{\alpha_{n+1}}.
\end{aligned}
\tag{21}
$$

Proof. Given ζ_{n+1} with $\operatorname{Im}\zeta_{n+1} \geq D_1/\alpha_{n+1}$, there is an integer i with $\frac{-1}{\alpha_{n+1}} \leq i \leq \frac{1}{\alpha_{n+1}}$ such that $\operatorname{Re} L_{n+1}^{-1}(\zeta_{n+1}+i) \in [\frac{1}{2\alpha_{n+1}}, \frac{1}{2\alpha_{n+1}}+2]$. By part (4) of Lemma 3.2, and Lemma 3.5, both with $r = 1/4$, we obtain

$$\operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}+i) \geq \operatorname{Im}(\zeta_{n+1}+i) - 4C_7 - C_4(1+\log\frac{1}{\alpha_{n+1}}).$$

By our assumption on $\operatorname{Im}\zeta_{n+1}$, this implies that

$$\operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}+i) \geq \frac{1}{4\alpha_{n+1}} + C_2.$$

Now, one uses part (1) of Lemma 3.2, to conclude that i iterates of $L_{n+1}^{-1}(\zeta_{n+1}+i)$ under F_{n+1} stay in $\Theta(C_2)$, and moreover,

$$\begin{aligned}\operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}) &= \operatorname{Im} F_{n+1}^{-i}(L_{n+1}^{-1}(\zeta_{n+1}+i))\\ &\geq \operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}+i) - \frac{i}{4}\\ &\geq \operatorname{Im}\zeta_{n+1} - 4C_7 - C_4(1+\log\frac{1}{\alpha_{n+1}}) - \frac{1}{4\alpha_{n+1}}.\end{aligned}$$

Using Lemma 3.1 with $r = 1/4$ at $L_{n+1}^{-1}(\zeta_{n+1})$ implies that

$$|\tau_{n+1}(L_{n+1}^{-1}(\zeta_{n+1}))| \leq 4C_1\alpha_{n+1}e^{-2\pi\alpha_{n+1}\left(\operatorname{Im}\zeta_{n+1}-4C_7-C_4(1+\log\frac{1}{\alpha_{n+1}})-\frac{1}{4\alpha_{n+1}}\right)}.$$

Hence, $\Phi_{n+1}(w_{n+1}) = \zeta_{n+1}$ implies

$$\begin{aligned}|w_{n+1}| &= |\Phi_{n+1}^{-1}(\zeta_{n+1})|\\ &\leq 4C_1e^{2\pi\alpha_{n+1}(+4C_7+C_4(1+\log\frac{1}{\alpha_{n+1}})+\frac{1}{4\alpha_{n+1}})}\alpha_{n+1}e^{-2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1}}\\ &\leq C\alpha_{n+1}e^{-2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1}},\end{aligned}$$

for some constant C.

As w_{n+1} is mapped to z_{n+1} in a bounded number of iterates $\sigma(n)$ under f_{n+1} which belongs to a compact class, $|z_{n+1}| \leq C'|w_{n+1}|$ for some constant C'. Therefore,

$$\begin{aligned}4/27e^{-2\pi\operatorname{Im}\zeta_n} &= |-4/27e^{-2\pi i\zeta_n}|\\ &= |z_{n+1}| \leq CC'\alpha_{n+1}e^{-2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1}}.\end{aligned}$$

Multiplying by 27/4 and then taking log of both sides, one obtains Inequality (21) for some constant D_2. □

Proof of Lemma 4.3. Given integer $\ell \geq 1$, we will show that there exists $m \geq \ell$ satisfying the inequality in the lemma. For arbitrary α, one of the following two occurs

($*$) There exists a positive integer $n_0 \geq \ell$ such that for every $j \geq n_0$, we have $\operatorname{Im}\zeta_j \geq \frac{D_1}{\alpha_j}$.

($**$) There are infinitely many integers j, $j \geq \ell$, with $\operatorname{Im}\zeta_j < \frac{D_1}{\alpha_j}$.

Assume conclusion of the lemma is not correct, that is, for every m greater than or equal to ℓ we have $\operatorname{Im}\zeta_m > \frac{1}{2\pi}\log\frac{1}{\alpha_{m+1}}$. We will show that each of the above cases leads to a contradiction.

If $(*)$ holds, we can use Lemma 4.4 for every $j \geq n_0$. So, for every integer n bigger than n_0, using Relation (21) repeatedly, we obtain

$$\begin{aligned}
\text{(22)} \quad \operatorname{Im}\zeta_n \leq{}& \frac{1}{\alpha_n\alpha_{n-1}\cdots\alpha_{n_0}}\operatorname{Im}\zeta_{n_0-1} - \frac{1}{2\pi\alpha_n\alpha_{n-1}\cdots\alpha_{n_0}}\log\frac{1}{\alpha_{n_0}} \\
&- \frac{1}{2\pi\alpha_n\alpha_{n-1}\cdots\alpha_{n_0+1}}\log\frac{1}{\alpha_{n_0+1}}\cdots - \frac{1}{2\pi\alpha_n}\log\frac{1}{\alpha_n} \\
&+ D_2\left(\frac{1}{\alpha_n\alpha_{n-1}\cdots\alpha_{n_0}} + \frac{1}{\alpha_n\alpha_{n-1}\cdots\alpha_{n_0-1}} + \cdots + \frac{1}{\alpha_n}\right).
\end{aligned}$$

Let $\beta_{-1} := 1$, and $\beta_j := \alpha_0\alpha_1\cdots\alpha_j$, for every $j \geq 0$. Using our contradiction assumption and then multiplying both sides of the above inequality by $2\pi\beta_n$, we see

$$\begin{aligned}
\sum_{j=n_0-1}^{n}\beta_j\log\frac{1}{\alpha_{j+1}} &\leq 2\pi\beta_{n_0-1}\operatorname{Im}\zeta_{n_0-1} + 2\pi D_2\left(\beta_{n_0-1} + \beta_{n_0} + \cdots + \beta_{n-1}\right) \\
&\leq 2\pi\operatorname{Im}\zeta_{n_0-1} + 2\pi D_2.
\end{aligned}$$

Since n was an arbitrary integer, this contradicts α being a non-Brjuno number.

Now assume $(**)$ holds. Let $n_1 < m_2 \leq n_2 < m_3 \leq n_3 < \cdots$ be an increasing sequence of positive integers with the following properties

- For every integer j with $m_i \leq j \leq n_i$, we have $\operatorname{Im}\zeta_j < \frac{D_1}{\alpha_j}$
- For every integer j with $n_i < j < m_{i+1}$, we have $\operatorname{Im}\zeta_j \geq \frac{D_1}{\alpha_j}$.

Estimate (22) holds for $j = n_i+1, n_i+2, \cdots, m_{i+1}-1$, where Lemma 4.4 can be used, and implies that for every $i \geq 2$:

$$\sum_{j=n_i}^{m_{i+1}-1}\beta_j\log\frac{1}{\alpha_{j+1}} \leq 2\pi\beta_{n_i}\operatorname{Im}\zeta_{n_i} + 2\pi D_2\left(\beta_{n_i} + \beta_{n_i+1} + \cdots + \beta_{m_{i+1}-2}\right).$$

Hence,

$$\begin{aligned}
\sum_{j=m_2}^{\infty}\beta_j\log\frac{1}{\alpha_{j+1}} &= \sum_{j;\, m_i\leq j<n_i}\beta_j\log\frac{1}{\alpha_{j+1}} + \sum_{j;\, n_i\leq j<m_{i+1}}\beta_j\log\frac{1}{\alpha_{j+1}} \\
&\leq \sum_{j;\, m_i\leq j<n_i}\beta_j\log\frac{1}{\alpha_{j+1}} + 2\pi\sum_{i=2}^{\infty}\beta_{n_i}\operatorname{Im}\zeta_{n_i} + 2\pi D_2\sum_{j;\, n_i\leq j<m_{i+1}-1}\beta_j.
\end{aligned}$$

In the first and the second sums we have used $\frac{1}{2\pi}\log\frac{1}{\alpha_{j+1}} < \operatorname{Im}\zeta_j < \frac{D_1}{\alpha_j}$. Therefore, the whole sum is less than

$$2\pi D_1 \sum_{j;\, m_i\le j<n_i} \beta_{j-1} + 2\pi D_1 \sum_{i=2}^{\infty} \beta_{n_i-1} + 2\pi D_2 \sum_{j=n_2}^{\infty} \beta_j$$

which contradicts α being a non-Brjuno number. $\square$

4.3. **Going up the renormalization tower.** Recall the sectors $\mathcal{C}_n^{-i} \cup (\mathcal{C}_n^\sharp)^{-i}$, for $i = 1, 2, \dots, k_n$, introduced in definition of the renormalization (for f_n), where $S_n^0 = \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n}$. If $k_n < k'+1$, by our assumption (17) on k', we can consider further pre-images for $i = k_n+1, \cdots, k'+1$ as

$$\begin{aligned}\mathcal{C}_n^{-i} &:= \Phi_n^{-1}(\Phi_n(\mathcal{C}_n^{-k_n}) - (i-k_n)),\\ (\mathcal{C}_n^\sharp)^{-i} &:= \Phi_n^{-1}(\Phi_n((\mathcal{C}_n^\sharp)^{-k_n}) - (i-k_n)).\end{aligned}$$

Let $\mathcal{D}_n$ denote the sector $\mathcal{C}_n^{-k'-1} \cup (\mathcal{C}_n^\sharp)^{-k'-1}$, and observe that $f_n^{k'+1} : \mathcal{D}_n \to f_n^{k_n}(S_n^0)$.

For every integer $n \ge 0$, define the set $\mathcal{P}_n^\natural$ as:

$$\mathcal{P}_n^\natural := \bigcup_{j=0}^{k'} f_n^j(\mathcal{D}_n).$$

We define a map $\Phi_n^\natural : \mathcal{P}_n^\natural \to \mathbb{C}$, using the dynamics of f_n, as follows. For $z \in \mathcal{P}_n^\natural$, there is an integer j with $0 \le j \le k'+1$, such that $f_n^j(z) \in \mathcal{P}_n$. Now, let

$$\Phi_n^\natural(z) := \Phi_n(f_n^j(z)) - j.$$

As Φ_n satisfies the Abel functional equation, one can see that $\Phi_n^\natural$ matches on the boundary of above sectors and gives a well defined holomorphic map on $\mathcal{P}_n^\natural$. The map $\Phi_n^\natural$ is not univalent, however, it still satisfies the Abel Functional equation on $\mathcal{P}_n^\natural$. It has critical points at the critical point of f_n and its pre-images within $\mathcal{P}_n^\natural$. The $k'+1$ critical points of $\Phi_n^\natural$ are mapped to $0, 1, \dots, k'$.

The map $\Phi_n^\natural$ is a natural extension of Φ_n to a multi-valued holomorphic map on $\mathcal{P}_n^\natural \cup \mathcal{P}_n$. However, the two maps

$$\Phi_n^\natural : \bigcup_{j=0}^{k'} f_n^j(\mathcal{D}_n) \to \mathbb{C}, \quad \text{and} \quad \Phi_n : \bigcup_{j=k'+1}^{1/\alpha_n+k'-k-1} f_n^j(\mathcal{D}_n) \to \mathbb{C}$$

provide a well-defined holomorphic map on every $k'+1$ consecutive sectors of the form $f_n^j(\mathcal{D}_n)$. We denote this map by $\Phi_n^\natural \amalg \Phi_n$. More precisely, for every l with $0 \le l < \frac{1}{\alpha_n} - k$,

$$\Phi_n^\natural \amalg \Phi_n : \bigcup_{j=0}^{k'} f_n^{l+j}(\mathcal{D}_n) \to \mathbb{C}$$

is defined as

$$\Phi_n^\natural \amalg \Phi_n = \begin{cases} \Phi_n^\natural(z), & \text{if } z \in f_n^i(\mathcal{D}_n) \text{ and } i < k'+1; \\ \Phi_n(z), & \text{if } z \in f_n^i(\mathcal{D}_n) \text{ and } i \geq k'+1. \end{cases}$$

Consider the Sequence (18) and assume that $\operatorname{Im}\zeta_n \leq \frac{1}{2\pi}\log\frac{1}{\alpha_{n+1}}$ holds for some positive integer n. Let $\mathcal{A}_n$ denote the topological disk $B_{\delta_1}(B(\gamma_n(1), r^*) \cup \gamma_n[0,1]$ with γ_n and r^* were introduced in Lemma 4.1. We will define domains $V_n, V_{n-1}, \dots, V_1$ and holomorphic maps $g_{n+1}, g_n, \dots, g_1$ satisfying the following diagram

$$\mathcal{A}_n \xrightarrow{g_{n+1}} V_n \xrightarrow{g_n} V_{n-1} \xrightarrow{g_{n-1}} \cdots V_1 \xrightarrow{g_1} V_0 := B(\Omega_0^0, 1) \tag{23}$$

where,

$$\begin{aligned} & V_m = \Omega_m^0 \setminus I_{m,j(m)}, \text{ for some } j(m) \text{ with } 0 \leq j(m) < \tfrac{1}{\alpha_m} - k, \\ & g_{n+1} : \mathcal{A}_n \to V_n, g_1 : V_1 \to \Omega_0^0, \\ & g_m : V_m \to V_{m-1}, \text{ for every } m \text{ with } 1 \leq m \leq n, \\ & g_m(z_m) = z_{m-1} \text{ for every } m = 1, 2, \dots, n. \end{aligned} \tag{24}$$

The idea is to use an inductive process to define pairs $g_i + 1, V_i$, starting with $i = n$ and ending with $i = 0$.

Base step $i = n$**:** We have $\zeta_n \in \mathcal{A}_n$ and satisfies (20). As $\operatorname{diam}(\mathcal{A}_n) \leq 1 - \delta_1$, and $\delta_2 < \delta_1$, there exists an integer j, with $0 \leq j \leq 1$, such that

$$\operatorname{Re}(\mathcal{A}_n - j) \subset (0, \frac{1}{\alpha_n} - k).$$

We define $g_{n+1} : \mathcal{A}_n \to \mathbb{C}$ as

$$g_{n+1}(\zeta) := f_n^{j+\sigma(n)}(\Phi_n^{-1}(\zeta - j)).$$

By Lemma 4.1, $\mathbb{E}\mathrm{xp}(\mathcal{A}_n - j)$ is contained in Dom f_{n+1}. So Lemma 2.1 implies that $f_n^{j+\sigma(n)}$ is defined at every point in $\Phi_n^{-1}(\mathcal{A}_n - j)$, that is, the above map is well defined. Using the same lemma, as $j + \sigma(n) \leq 1 + k_n + k'$ by (19), we conclude that $g_n(\mathcal{A}_n)$ is contained in Ω_n^0.

Because $\mathcal{A}_n - j$ is contained in int $\Phi_n(\mathcal{P}_n)$, it does not intersect the vertical line $\{\xi \in \mathbb{C} : \operatorname{Re}\xi = 0\}$. Therefore, $\Phi_n^{-1}(\mathcal{A}_n - j)$ does not intersect the curve $I_{n,0}$. One can see from this that, $g_{n+1}(\mathcal{A}_n) = f_n^{j+\sigma(n)}(\Phi_n^{-1}(\mathcal{A}_n - j))$ does not intersect the curve $I_{n,j'}$, where j' is $j + \sigma(n)$ module $\lfloor\frac{1}{\alpha_n}\rfloor - k$. We define V_n as $\Omega_n^0 \setminus I_{n,j'}$.

Finally, by equivariance property of Φ_n,

$$g_{n+1}(\zeta_n) = f_n^{j+\sigma(n)}(\Phi_n^{-1}(\zeta_n - j)) = f_n^{\sigma(n)}(w_n) = z_n.$$

Induction step: Assume (g_{i+1}, V_i) is defined and we want to define (g_i, V_{i-1}). Since every closed loop in V_i is contractible in $\mathbb{C}^*$, there exists an inverse branch of $\mathbb{E}\mathrm{xp}$, denoted by η_i, defined on V_i with $\eta_i(z_i) = \zeta_{i-1}$. Now consider the following two cases,

*case i***:** $\operatorname{Re}(\eta_i(V_i)) \subset (1/2, \infty)$,

*case ii***:** $\operatorname{Re}(\eta_i(V_i)) \cap (-\infty, 1/2] \neq \varnothing$.

Case i. Since $\zeta_n \in \eta_i(V_i)$ satisfies (20) and diam $B_{\delta_2}(\eta_i(V_i)) \leq k'+1/2$ by (16), there exists an integer j, $0 \leq j \leq k'+1$, with

$$B_{\delta_2}(\eta_i(V_i)) - j \subset \{\xi \in \mathbb{C} : \tfrac{1}{4} \leq \operatorname{Re} \xi \leq \tfrac{1}{\alpha_{i-1}} - k - \tfrac{1}{2}\}. \tag{25}$$

We define $\widetilde{g}_i : B_{\delta_2}(\eta_i(V_i)) \to \mathbb{C}$ as

$$\widetilde{g}_i(\zeta) := f_{i-1}^{j+\sigma(i-1)}(\Phi_{i-1}^{-1}(\zeta - j)), \tag{26}$$

and let

$$g_i(z) := \widetilde{g}_i \circ \eta(z).$$

By Lemma 4.2, $\mathbb{E}\mathrm{xp}(B_{\delta_2}(\eta_i(V_i)))$ is contained in Dom f_i. Thus, Lemma 2.1 and condition (17) implies that $f_{i-1}^{j+\sigma(i-1)}$ is defined on $\Phi_{i-1}^{-1}(B_{\delta_2}(\eta_i(V_i)) - j)$, that is, the above map is well defined.

By equivariance property of Φ_{i-1} (Theorem 1.3), we have

$$g_i(z_i) = f_{i-1}^{j+\sigma(i-1)}(\Phi_{i-1}^{-1}(\eta_i(z_{i-1}) - j)) = f_{i-1}^{\sigma(i-1)}(w_{i-1}) = z_{i-1}.$$

One also concludes from (25) that $B_{\delta_2}(\eta_i(V_i)) - j$ does not intersect the vertical line $\{\xi \in \mathbb{C} : \operatorname{Re} \xi = 0\}$. Therefore,

$$\widetilde{g}_i(B_{\delta_2}(\eta(V_i)) - j) = f_{i-1}^{j+\sigma(i-1)}(\Phi_{i-1}^{-1}(B_{\delta_2}(\eta_i(V_i)) - j))$$

does not intersect the curve $I_{i-1,j'}$, where j' is $j + \sigma(i-1)$ module $\lfloor \frac{1}{\alpha_{i-1}} \rfloor - k$. Hence, by defining $V_{i-1} := \Omega_{i-1}^0 \setminus I_{i-1,j'}$, we have

$$\widetilde{g}_i(B_{\delta_2}(\eta_i(V_i)) - j) \subset V_{i-1}, \tag{27}$$

which will be used later.

Case ii. Because diam $(\eta_i(V_i)) \leq k'$ and $\eta_i(V_i)$ contains ζ_{i-1} which satisfies (20), we must have $\zeta_{i-1} \in B(j, \delta_2)$ for some j in $\{1, 2, \ldots, k'\}$.

We claim that

$$B_{\delta_2}(\eta_i(V_i)) \cap \{0, -1, -2, \ldots, -k'\} = \varnothing. \tag{28}$$

As $\mathbb{E}\mathrm{xp}(\mathbb{Z}) = -4/27$, it is equivalent to see that $\eta_i(-4/27) \notin \{0, -1, \ldots, -k'\}$. But by inclusion in the Lemma (4.2), for every integer n, we have

$$\mathbb{E}\mathrm{xp}(B(n, \delta_2)) \subset \operatorname{int} V_i.$$

This implies that $\eta_i(-4/27) \in \{1, 2, \cdots, k'\}$.

The set $B_{\delta_2}(\eta_i(V_i))$ has diameter strictly less than $k'+1$, so it can intersect at most $k'+1$ vertical strips of width 1. More precisely, $B_{\delta_2}(\eta_i(V_i))$ is contained in the $k'+1$ consecutive sets in the list

$$\Phi_{i-1}^{\natural}(\mathcal{D}_{i-1}), \Phi_{i-1}^{\natural}(f_{i-1}(\mathcal{D}_{i-1})), \ldots, \Phi_{i-1}^{\natural}(f_{i-1}^{k'-1}(\mathcal{D}_{i-1})),$$
$$\Phi_{i-1}(f_{i-1}^{k'}(\mathcal{D}_{i-1})), \ldots, \Phi_{i-1}(f_{i-1}^{2k'+1}(\mathcal{D}_{i-1})).$$

Thus, by the above argument about $\Phi^{\natural}_{i-1} \amalg \Phi_{i-1}$, and that every closed loop in $B_{\delta_2}(\eta_i(V_i))$ is contractible in the complement of the critical values of $\Phi^{\natural}_{i-1} \amalg \Phi_{i-1}$, there exists an inverse branch of this map, denoted by $\widetilde{g}_i$, defined on $B_{\delta_2}(\eta_i(V_i))$. We let

$$g_i(z) := \widetilde{g}_i(\eta_i(z)) : V_i \to \Omega^0_{i-1}.$$

In this case $\sigma(i-1) = 0$, $\Phi_{i-1}(w_{i-1}) = \zeta_{i-1}$, and $w_{i-1} = z_{i-1}$. So $g_i(z_i) = z_{i-1}$.

Like previous case, one can see that $\widetilde{g}_i(B_{\delta_2}(\eta_i(V_i)))$ does not intersect the curve $I_{i-1,j}$ for $j = \sup\{\mathrm{Re}(B_{\delta_2}(\eta_i(V_i)))\} + 1$. We can define $V_{i-1} := \Omega^0_{i-1}$ and obtain $g_i : V_i \to V_{i-1}$. Indeed, we have

$$\widetilde{g}_i(B_{\delta_2}(\eta_i(V_i))) \subset V_{i-1}. \tag{29}$$

This finishes definition of the domains and maps satisfying (24).

Each domain $V_n, V_{n-1} \dots, V_0$, is a hyperbolic Riemann surface. Let ρ_i denote the Poincaré metric on V_i, that is, $\rho_i(z)|dz|$ is the complete metric of constant negative curvature on V_i. Similarly, ρ_{n+1} denotes the Poincaré metric on $\mathcal{A}_n$. The following two lemmas are natural consequence of our construction of the chain (23).

Lemma 4.5. *Each map $g_i : (V_i, \rho_i) \to (V_{i-1}, \rho_{i-1})$, for $i = n, n-1, \dots, 1$, is uniformly contracting. More precisely, for every $z \in V_i$, we have*

$$\rho_{i-1}(g_i(z)) \cdot |g_i'(z)| \le \delta_3 \cdot \rho_i(z),$$

for $\delta_3 = \frac{2k'+1}{2k'+1+\delta_2}$.

Proof. Let $\tilde{\rho}_i(z)|dz|$ and $\hat{\rho}_i(z)|dz|$ denote the Poincaré metric on the domains $\eta_i(V_i)$ and $B_{\delta_2}(\eta_i(V_i))$, respectively. By definition of g_i and properties (27) and (29) we can decompose the map $g_i : (V_i, \rho_i) \to (V_{i-1}, \rho_{i-1})$ as follows:

$$(V_i, \rho_i) \xrightarrow{\eta_i} (\eta_i(V_i), \tilde{\rho}_i) \xhookrightarrow{inc.} (B_{\delta_2}(\eta_i(V_i)), \hat{\rho}_i) \xrightarrow{\tilde{g}_i} (V_{i-1}, \rho_{i-1}).$$

By Schwartz-Pick lemma the first map, and the last map are non-expanding, i.e.,

$$\tilde{\rho}_i(\eta_i(\zeta))\, |\eta_i'(\zeta)| \le \rho_i(\zeta), \text{ and } \rho_{i-1}(\tilde{g}_i(\zeta))\, |\tilde{g}_i{}'(\zeta)| \le \hat{\rho}_i(\zeta).$$

To show that the inclusion map is uniformly contracting in the respective metrics, fix an arbitrary point ζ_0 in $\eta_i(V_i)$, and define

$$H(\zeta) := \zeta + (\zeta - \zeta_0)\frac{\delta_2}{(\zeta - \zeta_0 + 2k' + 1)} : \eta_i(V_i) \to \mathbb{C}.$$

Since diam $\eta_i(V_i) \le k'$, we have $|\,\mathrm{Re}(\zeta - \zeta_0))| \le k'$ for every $\zeta \in \eta_i(V_i)$ and also $H(\zeta_0) = \zeta_0$. This implies that $|\frac{\zeta-\zeta_0}{\zeta-\zeta_0+2k'+1}| < 1$. Thus,

$$|H(\zeta) - \zeta| = \delta_2 |\frac{\zeta - \zeta_0}{\zeta - \zeta_0 + 2k' + 1}| < \delta_2,$$

which implies that $H(\zeta)$ is a holomorphic map from $\eta_i(V_i)$ into $B_{\delta_2}(\eta_i(V_i))$. By Schwartz-Pick lemma, H is non-expanding. In particular at ζ_0, we obtain

$$\tilde{\rho}_i(\zeta_0)|H'(\zeta_0)| = \tilde{\rho}_i(\zeta_0)(1 + \frac{\delta_2}{2k'+1}) \le \hat{\rho}_i(\zeta_0).$$

That is, $\hat{\rho}_i(\zeta_0) \le \delta_3 \cdot \tilde{\rho}_i(\zeta_0)$ for $\delta_3 = \frac{2k'+1}{2k'+1+\delta_2} < 1$. Putting all this together gives the inequality in the lemma. $\square$

Lemma 4.6. *There exists a positive constant δ_4 such that for every $i = 1, 2, \dots, n+1$, the following holds*

- *The map $g_i : V_i \to V_{i-1}$ is univalent or has only one simple critical point .*
- *The map $g_i : V_i \to V_{i-1}$ is univalent on the hyperbolic ball*

$$B_{\rho_i}(z_i, \delta_4) := \{z \in V_i \mid d_{\rho_i}(z, z_i) < \delta_4\}.$$

Proof. Each map g_i is composition of at most four maps; η_i, a translation by an integer j, Φ_{i-1}^{-1}, and $f_{i-1}^{j+\sigma(i-1)}$. The first three maps are univalent. The map $f_{i-1}^{j+\sigma(i-1)}$ is univalent or has at most one simple critical in $\Phi_{i-1}^{-1}(\eta_i(V_i) - j)$. To see this, first note that the critical points of $f_{i-1}^{j+\sigma(i-1)}$ are

$$\{\mathrm{cp}_{f_{i-1}}, f_{i-1}^{-1}(\mathrm{cp}_{f_{i-1}}), \cdots f_{i-1}^{-j-\sigma(i-1)}(\mathrm{cp}_{f_{i-1}})\}.$$

All of them are non-degenerate and, by our technical assumption, $j + \sigma(i-1) \le 2k'+1$. If $\Phi_{i-1}^{-1}(\eta_i(V_i) - j)$ contains more than one point in the above list, by equivariance property of Φ_{i-1}, there must be a pair of points ξ, $\xi + m$ (for some integer m) in $\eta_i(V_i) - j$. As this set is a lift of a simply connected domain in $\mathbb{C}^*$ under $\mathbb{E}\mathrm{xp}$, that is not possible.

The maps g_i introduced in *case ii* are univalent, therefore, to see the second part, we only need to consider maps introduced in *case i* in the above inductive process. First we claim that there exists a real constant $\delta > 0$, such that the ball

$$\{z \in V_{i-1} : d_{\rho_{i-1}}(z, z_{i-1}) < \delta\}$$

is simply connected and does not contain critical value of g_i (if there is any critical value).

Assuming the claim for a moment, one can take $\delta_4 := \delta$. Because, by the previous lemma, image of $B_{\rho_i}(z_i, \delta_4)$ is contained in the above ball. As $B_{\rho_{i-1}}(z_{i-1}, \delta)$ is simply connected and does not contain any critical value, one can find a univalent inverse branch for g_i on this ball. Therefore, g_i is univalent on the ball $B_{\rho_i}(z_i, \delta_4)$.

To prove the claim, note that by definition (26) of $\widetilde{g}_{i-1}$, and condition $0 \le j \le k'+1$, a possible critical value of g_{i-1} can only be one of

$$-4/27, f_{i-1}(-4/27), \dots, f_{i-1}^{k'}(-4/27).$$

First we show that if $\mathrm{cv}_{g_{i-1}}$ belongs to $\Phi_{i-1}^{-1}(B(l, \delta_2))$ for some $l \in \{1, 2, \dots, k'\}$, then z_{i-1} does not belong to this set. To see this, we consider *case I* and *case II* in the definition of quadruples (18) separately.

If *case I* holds, then we have $\sigma(i-1) = 0$, $z_{i-1} = w_{i-1}$, and $\zeta_{i-1} = \Phi_{i-1}(z_{i-1})$. If $\mathrm{Re}\,\zeta_{i-1} \ge k' + 1/2$, then z_{i-1} does not belong to any of the balls $\Phi_{i-1}^{-1}(B(l, \delta_2))$ for $l \in$

$\{1, 2, \ldots, k'\}$. If $\zeta_{i-1} \in B(l, \delta_2)$ for some $l \in \{1, 2 \ldots, k'\}$, then by (26) there is no critical value of g_{i-1} in any of $\Phi_{i-1}^{-1}(B(l, \delta_2))$.

If *case II* holds, then, by definition of the quadruples, z_{i-1} does not belong to any of $\Phi_{i-1}(B(l, \delta_2))$ for $l = 1, 2, \ldots, k'$.

Finally, we need to show that each set $\Phi_{i-1}^{-1}(B(l, \delta_2))$, for $l = 1, 2 \ldots, k'$, contains a hyperbolic ball of radius δ independent of l and i. Fix such an l, and observe that

$$\operatorname{mod}\big(\Phi_{i-1}(\mathcal{P}_{i-1}) \setminus \{B(l, \delta_2) \cup \{l - \mathbf{i}t \mid t \in [0, 2]\}\big) \geq c$$

for some constant $c > 0$. By Koebe distortion theorem for Φ_{i-1}^{-1}, we conclude that

$$\frac{\text{Euclidean diameter}\,(\Phi_{i-1}^{-1}(B(l, \delta_2)))}{\text{Euclidean distance between } \Phi_{i-1}^{-1}(l) \text{ and } \partial V_{i-1}} \geq c'.$$

As ρ_{i-1} is proportional to one over distance to the boundary in V_{i-1}, the set $\Phi_{i-1}^{-1}(B(l, \delta_2))$ contains a round hyperbolic ball of radius uniformly bounded below. It is clear that each of these balls is simply connected. □

Let $\mathcal{G}_n$ denote the map

$$\mathcal{G}_n := g_1 \circ g_2 \cdots \circ g_{n+1} : \mathcal{A}_n \to \Omega_0^0.$$

Recall that γ_n is the line segment obtained in Lemma 4.1, and $\gamma_n(0) = \zeta_n$. So, $\mathcal{G}_n(\gamma_n(0)) = z_0$. The following lemma guarantees that $\mathcal{G}_n$ safely transfers the ball from level $n+1$ to the dynamic plane of f_0.

Lemma 4.7. *There exists a positive constant D_3 such that for every $\mathcal{G}_n$ introduced above, there is a positive constant r_n with the following properties,*

(1) $\mathcal{G}_n(B(\gamma_n(1), r^*)) \cap \Omega_0^{n+1} = \varnothing$,
(2) $B(\mathcal{G}_n(\gamma_n(1)), r_n) \subset \mathcal{G}_n(B(\gamma_n(1), r^*))$, *and* $|\mathcal{G}_n(\gamma_n(1)) - z_0| \leq D_3 \cdot r_n$,
(3) $r_n \leq D_3(\delta_3)^n$.

Proof.

Part (1): By Lemma 4.1, for every $z \in B(\gamma_n(1), r^*)$ we have

$$\mathbb{E}\mathrm{xp}(z) \notin \Omega_{n+1}^0, \text{ and } f_{n+1}(\mathbb{E}\mathrm{xp}(z)) \notin \Omega_{n+1}^0.$$

We claim that this implies

$$g_{n+1}(z) \notin \Omega_n^1, \text{ and } f_n(g_{n+1}(z)) \notin \Omega_n^1,$$

where

$$\Omega_n^1 = \bigcup_{i=0}^{\lfloor 1/\alpha_n \rfloor (k_{n+1} + 1/\alpha_{n+1} - k - 1) + 1} f_n^i(\psi_{n+1}(S_{n+1}^0)).$$

That is because if $g_{n+1}(z) \in \Omega_n^1$, then by definition of renormalization and definition of Ω_n^1, there is $a \in \mathcal{P}_n \cap \Omega_n^1$, and $b \in \mathcal{P}_n \cap \Omega_n^1$, such that $f_n^{i_1}(a) = g_{n+1}(z)$, $f_n^{i_2}(g_{n+1}(z)) = b$, $\mathbb{E}\mathrm{xp}(\Phi_n(a)) = z$, and $\mathbb{E}\mathrm{xp}(\Phi_n(b)) = f_{n+1}(z)$ for non-negative integers i_1 and i_2. One can see from this that $\mathbb{E}\mathrm{xp}(\Phi_n(a)) = z \in \Omega_{n+1}^0$ and $\mathbb{E}\mathrm{xp}(\Phi_n(b) = f_{n+1}(z) \in \Omega_{n+1}^0$ which contradicts our assumption.

The same argument implies the following statement for every $i = n, n-1, \dots, 1$,

$$\text{If } w \notin \Omega_i^{n-i+1}, \text{ and } f_i(w) \notin \Omega_i^{n-i+1}$$
$$\text{then } g_i(w) \notin \Omega_{i-1}^{n-i+2}, \text{ and } f_{i-1}(g_i(w)) \notin \Omega_{i-1}^{n-i+2}$$

where Ω_l^k is defined accordingly. One infers from these, with an induction argument, that $\mathcal{G}_{n+1}(z) \notin \Omega_{n+1}^0$.

Part (2): It follows from part (4) of Lemma 4.1 that $B(\gamma_n(1), r^*) \cup \gamma_n[0,1]$ has hyperbolic diameter (with respect to ρ_{n+1} in $\mathcal{A}_n$) uniformly bounded by some constant C (independent of n). Let l be the smallest non-negative integer with

$$C \cdot (\delta_3)^l \leq \delta_4/2.$$

We decompose the map $\mathcal{G}_{n+1}$ into two maps

$$\mathcal{G}_{n+1}^1 := g_{n-l+1} \circ g_{n-l+2} \circ \cdots \circ g_{n+1} \text{ and } \mathcal{G}_{n+1}^2 := g_1 \circ g_{n-l+2} \circ \cdots \circ g_{n-l}.$$

By Lemma 4.5 and our choice of l, we have

$$\mathcal{G}_{n+1}^1(B(\gamma_n(1), r^*) \cup \gamma_n[0,1]) \subseteq B_{\rho_{n-l}}(z_{n-l}, \delta_4/2).$$

Since each map g_i is uniformly contracting and univalent on $B_{\rho_i}(z_i, \delta_4)$, by Lemmas 4.5 and 4.6, $\mathcal{G}_{n+1}^2$ is univalent on $B_{\rho_{n-l}}(z_{n-l}, \delta_4)$. Moreover, by Koebe distortion theorem, it has bounded distortion on $\mathcal{G}_{n+1}^1(B(\gamma_n(1), r^*) \cup \gamma_n[0,1])$.

We claim that $\mathcal{G}_{n+1}^1$ belongs to a compact class. That is because it is composition of l maps (uniformly bounded independent of n) g_i, for $i = n+1, \cdots, n-l+1$, where each of them is composition of three maps as

$$g_i = f_i^{\sigma(i)+j} \circ \tilde{g}_i \circ (\eta_i - j).$$

The map η_i is univalent on V_i and, by Koebe distortion theorem, has uniformly bounded distortion on sets of bounded hyperbolic diameter. The map $\tilde{g}_i$ extends over the larger set $B_\delta(\eta_i(V_i))$ (see Equations (27) and (29)), so it also has uniformly bounded distortion. Finally, $f_i^{\sigma(i)+j}$ is a finite iterate of a map in a compact class.

Putting all these together, one infers that Euclidean diameter of the domain $\mathcal{G}_{n+1}^1(B(\gamma_n(1), r^*))$ is proportional to the Euclidean distance between two points $\mathcal{G}_{n+1}^1(\gamma_n(1))$ and $z_0 = \mathcal{G}_{n+1}(\gamma_n(0))$. Similarly, $\mathcal{G}_{n+1}^1(B_{r^*})$ contains a round ball of Euclidean radius proportional to its diameter. By previous argument about $\mathcal{G}_{n+1}^2$, this finishes part (2) of the lemma.

Part (3): First observe that $\mathcal{G}_{n+1}(B(\gamma_n(1), r^*))$ is contained in the compact subset Ω_0^0 of V_0 where the Euclidean and the hyperbolic metrics are proportional. The uniform contraction in Lemma 4.5 with respect to the hyperbolic metric implies the statement in this part. $\square$

4.4. **Proof of the corollaries.**

Corollary 4.8. *There exists a positive integer N such that for every non-Brjuno number $\alpha \in Irr_N$ the post-critical set of P_α has area zero.*

Proof. If $\mathcal{PC}(P_\alpha) = \mathcal{PC}(f_0)$ has positive area, by Lebesgue density point theorem, for almost every point z in this set we must have

$$\lim_{r\to 0} \frac{area(B(z,r) \cap \mathcal{PC}(f_0))}{areaB(z,r)} = 1.$$

Let z_0 be an arbitrary point different from zero in $\mathcal{PC}(f_0)$. By Proposition 2.4, z_0 is contained in the intersection of Ω_0^n, for $n = 0, 1, \cdots$. Thus, we can define the sequence of Quadruples (18). Lemma 4.3 provides us with an strictly increasing sequence of integers n_i for which we have

$$\operatorname{Im} \zeta_{n_i} \le \frac{1}{2\pi} \log \frac{1}{\alpha_{n_i+1}}.$$

With Lemma 4.1 at levels n_i, we obtain curves γ_{n_i} and balls $B(\gamma_{n_i}(1), r^*)$ enjoying the properties in that lemma. One introduces the sequence $\mathcal{G}_{n_i+1}$ which, by Lemma 4.7, provides us with a sequence of balls $B(\gamma_{n_i}(1), r_{n_i})$ satisfying

$$B(\gamma_{n_i}(1), r_{n_i}) \cap \Omega^0_{n_i+1} = \varnothing,\ |\mathcal{G}_{n_i+1}(\gamma_{n_i}(1)) - z_0| \le D_3 \cdot r_{n_i},\ \text{and } r_{n_i} \to 0.$$

With $s_i := r_{n_i} + D_3 \cdot r_{n_i}$ we have

$$\begin{aligned}\frac{\operatorname{area}\left(B(z_0, s_i)\right) \cap \mathcal{PC}(f_0))}{\operatorname{area}\left(B(z_0, s_i)\right)} &\le \frac{\pi(s_i)^2 - \pi(r_{n_i})^2}{\pi(s_i)^2}\\ &\le \frac{(D_3)^2 + 2D_3}{(D_3)^2 + 2D_3 + 1}\\ &< 1.\end{aligned}$$

which contradicts z_0 being a Lebesgue density point of $\mathcal{PC}(f_0)$. □

Corollary 4.9. *There exists a positive integer N such that, if $\alpha \in Irr_N$ is a non-Brjuno number, then Lebesgue almost every point in the Julia set of P_α is non-recurrent.*

In particular, there is no finite absolutely continuous (with respect to the Lebesgue measure) invariant measure supported on the Julia set.

Proof. By Propositions 2.4 and 1.2 almost every point in the complement of Ω_0^n is non-recurrent. As area Ω_0^n shrinks to zero, we conclude the first part in the lemma. The second part follows from the first part and the Poincaré recurrence theorem. □

Corollary 4.10. *There exists a positive integer N such that for every non-Brjuno number $\alpha \in Irr_N$ the post-critical set of P_α is connected.*

Proof. We claim that $\mathcal{PC}(P_\alpha) = \bigcap_{n=0}^{\infty} \Omega_n^0$. As each set Ω_n^0 is connected and intersection of a nest of connected sets is connected, the corollary follows from this claim.

To prove the claim, let $z \neq 0$ be an arbitrary point in the above intersection. We can build the sequence of Quadruples (18) corresponding to z. By Lemma 4.3, there is an

infinite sequence of positive integers n_i satisfying $\operatorname{Im}\zeta_{n_i} \leq \frac{1}{2\pi}\log\frac{1}{\alpha_{n_i+1}}$. It follows from proof of Lemma 4.1 that there exists a point ζ'_{n_i} in the lift $\eta_{n_i}(\Phi_{n_i}^{-1}(\lfloor\frac{1}{2\alpha_{n_i}}\rfloor))$ such that, $|\operatorname{Re}(\zeta_{n_i}-\zeta'_{n_i})| \leq 1/2$, and $|\zeta_{n_i}-\zeta'_{n_i}|$ is uniformly bounded. One transfers these two point to the dynamic plane of f_0 by $\mathcal{G}_{n_i+1}$ and concludes from Lemma 4.5 that $|\mathcal{G}_{n_i+1}(\zeta'_{n_i})-z|$ goes to zero as n_i tends to infinity. The point $\Phi_{n_i+1}^{-1}(\lfloor\frac{1}{2\alpha_{n_i+1}})\rfloor)$ belongs to the orbit of critical value of f_{n_i+1}, therefore by definition of renormalization, see Lemma 2.1, $\mathcal{G}_{n_i+1}(\zeta'_{n_i+1})$ belongs to the orbit of the critical value of f_0. Thus, $\mathcal{G}_{n_i+1}(\zeta'_{n_i+1}) \in \mathcal{PC}(f_0)$ and converges to z. This finishes proof of the claim. □

A corollary of the above proof is the following:

Corollary 4.11. *There are positive constants M, N, and $\mu < 1$ such that for every $\alpha \in Irr_N$ and every $z \in \Omega_0^{n+1}$ we have*

$$\| P_\alpha^{q_n}(z) - z \| \leq M\mu^n.$$

In particular this holds on the post-critical set.

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